SELF-INJECTIVE RINGS

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ABSTRACT. In 1958 Matlis proved that the study of Noetherian complete
local rings could be subsumed under the study of injective modules $E$ over a
commutative ring $A$ such that $B = \text{End}_A E$ is commutative. In this case
$B = \text{End}_B E$, and $E$ is said to be strongly balanced over $B$. The main
theorem of this paper shows that the study of strongly balanced injectives
over any ring, and hence the study of Morita self-dualities, is contained in
the study of self-injective rings.

Introduction. Let $\text{mod-}A$ ($A$-mod) denote the category of all right (left)
$A$-modules over a ring $A$. For a noncommutative ring $B$ and a two-sided
$B$-bimodule $E$, in a natural way the Cartesian product $R$ is a ring, the
so-called split-null or trivial extension of $E$ by $B$; also called the semidirect
product (ring) of the bimodule $E$ and denoted by $R = (B, E)$.

Theorem. $R = (B, E)$ is an injective (injective cogenerator) in $\text{mod-}R$ iff $E$
is an injective (injective cogenerator) in $\text{mod-}B$ such that $B = \text{End}_B E$
canonically.

This theorem shows that any ring $B$ with an injective bimodule $E$ such that
$B = \text{End}_B E$ is isomorphic to a factor ring $R/(0, E)$ of a self-injective ring,
and also leads to new examples of self-injective rings, notably those which are
not injective cogenerator (= PF) rings, or not valuation rings.

When $R$, or $E$, is a two-sided injective cogenerator, the theorem is a
corollary of a theorem of Müller [23].

Propositions. We begin with the main lemma used in the proof of
Theorem 2.

1. Lemma. Let $R$ be a ring, let $E$ be an ideal which is its own left annihilator,
$\perp E = \{a \in R|ae = 0\} = E$, let $B = R/E$. Then $E$ is canonically a $B$-bimod-
ule. If

(1.1) $E$ is injective as a (canonical) right $B$-module, and
(1.2) $B \approx \text{End}_B E$ canonically,

then $R$ is right self-injective (= injective in $\text{mod-}R$).

Conversely, if $R$ is right self-injective, then for any ideal $A$, the left annihilator

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1 Part of this paper was written while the author was a visitor at the Institute for Advanced
Study.
$^\perp A$ is an injective right $R/A$-module, and $\text{End}^\perp A_R^A \approx R/\perp A$ canonically. Thus, in this case, any ideal $E$ satisfying $E = ^\perp E$ satisfies (1.1) and (1.2).

**Proof.** Let $F$ be the injective hull of $R$ in mod-$R$, and let

$$F_1 = \text{ann}_R E = \{ x \in F | xa = 0, \forall a \in E \}.$$ 

Then, $F_1$ is a right $B$-module, and $E = ^\perp E$ is an injective right $B$-module by (1.1). Since every $B$-submodule of $F_1$ is an $R$-submodule, then $F_1$ is an essential extension of $F_1 \cap R = E$ as an $R$-module, hence as a $B$-module, so injectivity of $E$ in mod-$B$ implies that $F_1 = \text{ann}_R E = E$. Thus, if $y \in F$, then $yE \subseteq \text{ann}_R E = E$, so $y$ induces an endomorphism $b \in B' = \text{End} E_R = \text{End} E_B$. Now every $r \in R$ induces an endomorphism $r \in \text{End} E_B$ via left multiplication; hence $B = R/\perp E = R/E$ embeds in $B'$ canonically. Since $B \approx B'$ canonically by the assumption (1.2), there exists $r \in R$ such that

$$yx = b(x) = r_x = rx, \ \forall x \in E,$$

so

$$(y - r)x = 0, \ \forall x \in E;$$

hence

$$y - r = c \in \text{ann}_R E = E \subseteq R.$$ 

Therefore, $y = r + c \in R$, $\forall y \in F$, proving that $F = R$ is injective. In this case, for any ideal $A$, $^\perp A$ is an injective right $R/A$-module (e.g., [3b, p. 66, Proposition 12]) and every $b \in \text{End} A_R^A$ is induced by an element $r \in R$; hence $R/\perp A \approx \text{End} A_R$. Also, $R/\perp A \approx \text{End}^\perp A_R = \text{End}^\perp A_R^A$, canonically. Taking $A = E = ^\perp E$, we have the stated properties (1) and (2).

2. **Theorem.** Let $R = (B, E)$ be the semidirect product of a bimodule $E$ over a ring $B$. Thus, $a(xb) = (ax)b$ for all $a, b \in B$ and $x \in E$, and in $R = B \times E$ addition is componentwise, and multiplication is defined by:

$$ (a, x)(b, y) = (ab, ay + xb).$$

(The ring $R$ is $\approx$ the ring of all $2 \times 2$ matrices $(\begin{smallmatrix} a & x \\ 0 & b \end{smallmatrix})$, with $a \in B, x \in E$, under ordinary matrix operations.) Then:

(2.2) $R$ is right self-injective iff $E$ is injective in mod-$B$, and $B = \text{End} E_B$ canonically.

(2.3) $R$ is a right injective cogenerator in mod-$R$ (= $R$ is right PF) iff $E$ is an injective cogenerator of mod-$B$ satisfying $B = \text{End} E_B$ canonically.

(2.4) Assuming (2.3), then $R$ is left PF iff $E$ is an injective cogenerator of $B$-mod, and $B = \text{End}_B E$ canonically.

**Proof.** (2.2). Identify $E$ with $E_1 = \{(0, x) | x \in E\}$ in $R$, and $B$ with $B_1 = \{(b, 0) | b \in B\}$. Clearly, $B \approx B_1 \approx R/E_1$ (under $b \mapsto (b, 0) \leftrightarrow (b, 0) + E_1$), and $^\perp E_1$ in $R$ is $E_1$ if $E$ is a faithful left $B$-module. Thus, assuming $E_B$ injective and $B = \text{End} E_B$, that is, assuming (1.1) and (1.2), we have $R$ is injective by Lemma 1. The converse also comes from Lemma 1.
(2.3). Assume that $R$ is right PF (= pseudo-Frobenius). By [3a, p. 148, 3.31], an injective right $R$-module $E$ is cogenerating iff every simple right $R$-module embeds in $E$. Since $R$ is a right injective cogenerator ring by assumption, every simple right $R$-module $V \hookrightarrow R$. Now, since $J = \text{rad } R$ contains any square-zero (or nilpotent or nil) ideal, then $J \supseteq E_1$; hence $R/J \approx B/\text{rad } B$, and every simple right $R$-module $V = R/M$ corresponds to a simple right $B$-module $V' = B/M'$. Since $V$ embeds in $R$, then $V'$ embeds in $R$. If $v \in R$ and $v = (b, x) \neq 0$ generates $V$, then $b = 0 \Rightarrow V \subseteq E$, and $b \neq 0 \Rightarrow \exists (0, y) \neq 0 \in E$ such that $(b, x)(0, y) = (0, by) \neq 0 \in V \cap E$; hence $V \cap E = V \subseteq E$ in both cases. This proves that every simple $B$-module $V'$ embeds in $E$. Since $E$ is injective by (2.2), this proves that $E$ is cogenerating in $\text{mod-}B$. Moreover, $B = \text{End}_E E$ via (2.2).

These remarks also suffice for the converse of (2.3), since $E$ cogenerating means every simple $B$-module $V'$ embeds in $E$; hence every simple $R$-module $V$ embeds in $E$. Thus, if $E$ is an injective cogenerator in $\text{mod-}B$, and $B = \text{End}_E E$, then $R$ is injective by (2.2), hence cogenerating inasmuch as every simple right $R$-module $V$ embeds in $E_1 = (0, E) \subseteq R$.

**Proof of (2.4).** Let $R$ be left PF. Since $E$ is an injective cogenerator of $\text{mod-}B$ (by assumption (2.3)), then $E$ is faithful as a right $B$-module (see, e.g., [3a, p. 92, II4(a)]); hence $E_1^+ = E_1$ follows, so $E_1$ is an injective left $B$-module, where $B = R/E_1$, and it is easy to see that $E \approx E_1$ is actually an injective cogenerator of $B$-mod: If $V$ is a simple left $B$-module, then $V$ is a simple left $R$-module, so $V \subseteq R$. But $E_1V = 0$, since $V$ is a $B$-module, so $V \subseteq E_1^+ = E_1$ making $E_1$ a cogenerator of $B$-mod (cf. [3b, p. 199, Exercise 1]). Conversely, if $E$ is an injective cogenerator of $B$-mod, and $B = \text{End}_B E$, then by the right-left symmetry of Lemma 1 $R$ is left self-injective, hence cogenerating inasmuch as every simple left $B$-module $V$ embeds in $E_1 = (0, E) \subseteq R$.

**2A. Corollary.** Let $R = (B, E)$ be the semidirect product of a ring $B$ and $B$-bimodule $E$. Then: $R$ is cogenerating (both sides) iff $E$ is a strongly balanced injective cogenerator over $B$ (both sides). In this case $R$ is PF (both sides).

**Proof.** A ring $R$ is cogenerating on both sides iff $R$ is PF on both sides (see [10]). Therefore, Theorem 2 applies.

Since there exist rings which are right cogenerating but not injective (see e.g. [17]), then (2.3) shows that $E$ a strongly balanced cogenerator over $\text{mod-}B$ does not imply that $R = (B, E)$ is cogenerating. However, a theorem of Faith and Walker (e.g. [3b, p. 206, Proposition 24.9]) implies that any semilocal right cogenerating ring is injective. Moreover, if $E$ is strongly balanced and cogenerating on both sides, then every one-sided ideal of $R$ is an annihilator [22]. Note: by starting with, e.g., a self-injective ring $B = E$, one obtains another self-injective ring $R = (B, E)$ having $B$ as a factor ring, etc.
Every known example of a right PF ring is left PF. (See [4a], [4b] for the background of this problem.)

2B. COROLLARY. If every right PF ring is left PF, then a bimodule \( E \) over a ring \( B \) satisfies (2.3) iff it satisfies the left-right symmetry (2.3)'.

PROOF. This follows from (2.4).

Thus, the question is whether right PF \( \Rightarrow \) left PF can be reduced to a module-theoretic question. Conceivably a negative answer could be found for the latter for the case when \( E \) is some strongly balanced injective cogenerator in \( \text{mod-} B \) for an integral domain \( B \). Thus, does (2.3) imply the following three conditions?

\[
(2.3)' \equiv \begin{cases} 
(2.3a)' & E \text{ is injective in } B\text{-mod}, \\
(2.3b)' & E \text{ is a cogenerator in } B\text{-mod}, \\
(2.3c)' & B = \text{End}_B E \text{ canonically.}
\end{cases}
\]

A theorem of Kato [10] implies that a right PF ring is left PF iff it is left self-injective, and therefore it suffices to prove or disprove (2.3a)' and (2.3c)'. Moreover, a theorem of E. A. Walker and the author (see, e.g., [3b, p. 206, Proposition 24.9]) implies that any finitely generated projective cogenerator over a semilocal ring is injective. Thus, since a right PF ring is semiperfect hence semilocal, then (2.3b)' implies (2.3a)'; that is, it also suffices to prove or disprove (2.3b)' and (2.3c)'.

A mapping \( f: L \rightarrow E \) of a left ideal \( L \) of \( B \) into a \( B \)-module \( E \) is a Baer homomorphism if there exists \( m \in E \) such that \( f(x) = mx, \forall x \in L \). Then \( E \) is (FP)-injective in \( B\text{-mod} \) if every mapping \( f: L \rightarrow E \) from any (finitely generated) left ideal \( L \) is a Baer homomorphism. Any right PF ring is left FP-injective (a result which follows from the theorem of Jain [25] to the effect that \( R \) is left FP-injective iff every finitely presented right \( R \)-module is torsionless). Moreover, \( R = (B, E) \) is left FP-injective only if \( E \) is FP-injective in \( B\text{-mod} \), so we conclude that (2.3) implies the latter. Thus, (2.3) does imply some form of injectivity of \( E \) in \( B\text{-mod} \). Actually, left FP-injectivity of \( (B, E) \) also implies: (1) that \( E \) is finitely quasi-injective in \( B\text{-mod} \) in the sense of [26], (2) that the right ideals of \( B \) satisfy the double annihilator condition with respect to \( E \), and similarly, (3) that the right \( B \)-submodules \( X \) of \( E \) of the form \( X = Y + EK \) for a finitely generated right ideal \( K \) of \( B \), and finitely generated \( B \)-submodule of \( E \) in \( \text{mod-} B \), also satisfy the double annihilator condition with respect to \( B \). (It would be of obvious interest to characterize FP-injectivity of \( (B, E) \).)

3. COROLLARY. If \( E \) is a \( B \)-bimodule satisfying (2.3), then \( B \) is semiperfect, and \( E \) is a finite direct sum of indecomposable injectives. Therefore, there are only finitely many nonisomorphic simple \( B \)-modules, and \( E \) has finite socle.
Proof. Since $R = (B, E)$ is right PF, then $R$ is semiperfect, e.g., by Osofsky's theorem [17] (cf. [3b, p. 213, Theorem 24.32]), and the rest follows from this.

4. Theorem. Let $B$ be a commutative Noetherian ring with a strongly balanced injective module $E$. Then $B = \prod_{i=1}^{n} B_i$ is a finite product of complete local rings, and $E = \sum_{i=1}^{n} \bigoplus E_i$, where $E_i$ is the smallest injective cogenerator of $B_i$, $i = 1, \ldots, n$. Thus, $E$ is the smallest injective cogenerator of $B$.

Proof. Since $B$ is Noetherian, $E$ is a finite coproduct $E = \prod_{i=1}^{n} E_i$ of indecomposable injectives. Since each $E_i$ has local endomorphism ring, the finite Krull-Schmidt theorem holds, and so $B$ is a semilocal ring, idempotents lift modulo radical (see [3b, p. 45, 18.26]), $B = \prod_{i=1}^{n} B_i$, where $B_i = e_i E_i \approx \operatorname{End}_B E_i$ is a local ring, and $e_i^2 = e_i \in B$ is the projection idempotent, $i = 1, \ldots, n$. Hence, we may assume $E$ is indecomposable and $B$ local. By Matlis' theorem [13], in order that $B$ be complete it is necessary and sufficient to show that $E$ is the injective hull of $V = B/\text{rad } B$. By the Matlis-Utumi theorem, $J = \text{rad } B$ is the set of all $b$ such that $bx = 0$ for some $x \neq 0$. Since $J$ is f.g., and $E$ is uniform, then $W = \operatorname{ann}_E J \neq 0$. Thus, $W$ is an $R/J$-module, hence is semisimple (= a direct sum of simples), whence simple by uniformity, so $W \approx R/J \hookrightarrow E$. Then, $E$ is the injective hull of $V = R/J$, as required.

4A. Corollary. If $B = \operatorname{End}_B E$ is a commutative local ring with f.g. radical $J$, and $E$ injective, then $E = E(B/J)$ is the injective hull of $B/J$. So $E$ is a cogenerator in $\text{mod-}B$.

Proof. Same.

4B. Corollary. If the semidirect product ring $R = (B, E)$ of a Noetherian commutative ring $B$ and module $E$ is self-injective, then $R$ is an injective cogenerator, and a finite product of local injective cogenerators.

Proof. By Theorem 2, $B = \operatorname{End}_B E$ canonically, and $E$ is an injective $B$-module, so Theorem 3B applies, and the rest is easy.

An application of Theorem 2 and Matlis' theorem [13] yields:

4C. Theorem. If $B$ is a Noetherian local ring, and $E = E(B/\text{rad } B)$ the injective hull, then $R = (B, E)$ is injective iff $B$ is complete. (Then $R$ is PF.)

A ring $R$ is a right valuation ring (VR) iff the right ideals of $R$ are linearly ordered by inclusion. (A chain ring is a variant term for VR.)

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2 If $S$ is a commutative ring with duality, then there exists a (self) duality context $B F_S$ where $F$ is the minimal injective cogenerator (Theorem of B. J. Müller [23]; see also Vamos [20, Corollary 1.7]). When the radical of $S$ is finitely generated, then Corollary 4A shows that there is just one self-duality. The dualities for commutative $S$ are in 1-1 correspondence with ring automorphisms of $S$ of order $< 2$ (Morita [15]; cf. [3b, p. 199, 23.35]). For other dualities, consult [1], [3b], [7], [13]–[16], [20], [21], [23], [24].
5A. Proposition. A semidirect product ring \( R = (B, E) \) is a right VR iff \( B \) is a right VR, \( E \) is uniserial, and \( bE = E, \forall 0 \neq b \in B \).

Proof. If \( R \) is a right VR, then \( B \cong R/(0, E) \) is a right VR, and \( E \cong (0, E) \) is uniserial. If \( b \neq 0 \in B \), then \( (b, 0)R \not\subseteq (0, E) \); hence
\[
(b, 0)R = (bB, 0) + (0, bE) \supseteq (0, E),
\]
so \( bE = E \). The converse follows by reading up.

A VD is a domain which is a VR. For simplicity, from here on we shall assume that \( B \) whence \( R \) is commutative.

5B. Corollary. Let \( E \) be a faithful \( B \)-module. Then \( R = (B, E) \) is a VR iff \( B \) is a VD and \( E \) is a uniserial divisible \( B \)-module.

Proof. Immediate.

5C. Corollary. Let \( E \) be a torsion free module over a domain \( B \). Then \( R = (B, E) \) is a VR iff \( B \) is a VD and \( E \) is a uniserial injective \( B \)-module. In this case \( R \) is injective iff \( E \) is strongly balanced.

Proof. Any torsion free divisible module over a domain is injective, so apply the corollary. (Conversely, any injective module is divisible.) The last sentence follows from Theorem 2.

6A. Theorem. Let \( R = (B, E) \) be a semidirect product ring. The f. a. e.:

1. \( R \) is a PFVR (= a VR which is PF).
2. \( B \) is an almost maximal valuation domain (AMVD), \( E = E(B/\text{rad } B) \) is the injective hull of \( B/\text{rad } B \), and \( B = \text{End}_A E \).
3. \( B \) is a local domain such that \( E = E(B/\text{rad } B) \) is uniserial and strongly balanced.
4. \( B \) is an MVD and \( E = E(B/\text{rad } B) \) is strongly balanced.

Proof. By Gill's theorem [5], a local ring \( B \) is AMVR iff \( E(B/J) \) is uniserial, where \( J = \text{rad } B \). Thus, using Theorems 2 and 5A, (2) \( \iff \) (3) follows. Moreover, (1) \( \iff \) (3) by 5C and Corollary 4A, and (2) \( \iff \) (4) by a theorem of Vamos [19].

6B. Corollary. If \( B \) is a Noetherian local domain, and \( E = E(B/J) \), then the semidirect product ring \( R = (B, E) \) is an injective VR iff \( B \) is a complete discrete valuation domain. In this case \( R \) is PF.

Proof. Follows from 6A and Matlis' theorem [13] (since \( B \) is a Noetherian VD).

7. Example. A noncongerenating injective local ring. (Levy [11].) Let \( F \) be a field, \( x \) an indeterminate, and \( W \) the family of all well-ordered sets of nonnegative real numbers. Let \( A \) denote the ring of all formal power series \( \sum_{a \in w} c_a x^w \), where \( c_a \in F \) and \( w \in W \) with the usual addition and multiplication. The proper ideals of \( A \) are: the principal ideals.
(x^b) = \{x^u | u \in A \}

and the ideals

(x^{c>b}) = \{x^u | u \in A^* \cup \{0\}, c > b \}

where A^* = units of A. (In particular, rad A = (x^{>0}).) Levy [11] proved that every proper factor ring is self-injective. Now R = A/(x) does not contain a minimal ideal, hence R is injective but not PF. [This corrects a statement of p. 216 of [3b] to the effect that every proper factor ring of A is PF! If every factor ring of a ring R is PF (= R is CPF), then R must be Artinian. (See for example [3b, p. 238, Proposition 25.4.6A].) However, no factor ring R = A/I, where I \neq rad A, can be Artinian, since (rad A)^2 = (rad A) \Rightarrow (rad R)^2 = (rad R).

Any infinite product of self-injective rings is self-injective, but never PF since never semiperfect, yielding additional examples of noncogenerating injective rings.

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ADDED JULY 1978. H. Sekiyama informs me that [28] contains the characterization of when R = (B, E) is injective (Corollary 4.36). In [27] he characterizes i.a. when R is QF-3.

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