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HOPFIAN BOOLEAN ALGEBRAS OF POWER LESS THAN OR EQUAL TO CONTINUUM

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ABSTRACT. An infinite Boolean algebra (BA) is said to be hopfian if every onto endomorphism is one-to-one. It is shown that there are *no* countably infinite hopfian BA's. An atomic hopfian BA of power continuum is constructed.

We restrict our attention to infinite Boolean algebras $\mathfrak{A} = (A, +, \cdot, -, 0, 1)$, and use End $\mathfrak{A}$ for the endomorphism semigroup of $\mathfrak{A}$, At($\mathfrak{A}$) for the set of atoms of $\mathfrak{A}$, and Sg(T) for the subalgebra of $\mathfrak{A}$ generated by $T \subseteq A$. $\mathfrak{P}(\omega)$ denotes the power set of ω and $\mathfrak{C}(\omega)$, the field of finite-cofinite subsets of ω. We use MA(κ) to denote Martin's axiom: If $\mathfrak{P}$ is a partial ordering satisfying the countable antichain condition, and if $\mathfrak{F}$ is a collection of dense open subsets of $\mathfrak{P}$ with $|\mathfrak{F}| \leqslant \kappa$, then there is an $\mathfrak{F}$-generic filter on $\mathfrak{P}$. (See A_κ of Martin and Solovay [2].) Recall that MA($\aleph_0$) is a theorem in ZFC.

THEOREM 1. *Let $\mathfrak{A}$ be a BA of power $\kappa < 2^{\aleph_0}$ with infinitely many atoms and assume* MA(κ) *holds. Then there is $\mathfrak{B} \subseteq \mathfrak{A}$ such that $\mathfrak{B} \cong \mathfrak{C}(\omega)$ and there is an isomorphism* $^+$: End $\mathfrak{B} \to$ End $\mathfrak{A}$ *such that, for $h \in$ End $\mathfrak{B}$, h is one-to-one (onto) iff h^+ is one-to-one (onto, respectively).*

The following lemma, which is (1) of Theorem 3.6 of McKenzie and Monk [3], is proved using MA(κ).

LEMMA 2. *Under the hypothesis of Theorem* 1, *there is a $D \subseteq$* At($\mathfrak{A}$), *with $|D| = \aleph_0$ such that for every $a \in A$, the set $S(a) = \{d \in D | d \leqslant a\}$ is a finite or cofinite subset of D.*

PROOF OF THEOREM 1. Let $\mathfrak{B} = \mathrm{Sg}(D)$ and notice $\mathfrak{B} \cong \mathfrak{C}(\omega)$. Clearly, $I = \{a \in A | S(a)$ is finite$\}$ is a maximal ideal of $\mathfrak{A}$. For $a \in I$, there is a unique finite $S(a) \subseteq D$ and unique $t(a) \in A$ such that $a = \sum S(a) + t(a)$ and $d \cdot t(a) = 0$ for all $d \in D$.

Let $h \in$ End $\mathfrak{B}$. For each $a \in I$, define

$$f(a) = \sum \{h(d) | d \in S(a)\} + t(a).$$

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Define $h^+: \mathfrak{A} \to \mathfrak{A}$ by

$$h^+(a) = \begin{cases} f(a) & \text{if } a \in I, \\ -f(-a) & \text{if } a \notin I. \end{cases}$$

It is easy to check that h^+ is an endomorphism of $\mathfrak{A}$ satisfying Theorem 1. $\square$

THEOREM 3. *Let* $\kappa < 2^{\aleph_0}$ *and assume* MA(κ). *If* $\mathfrak{A}$ *is a BA of power* κ *with infinitely many atoms, then* $\mathfrak{A}$ *is not hopfian.*

PROOF. By Theorem 1, it suffices to note that $\mathfrak{C}(\omega)$ is not hopfian. In fact, it is well known that any finite-to-one map $t: \omega \to \omega$ (i.e., $t^{-1}(\{i\})$ is finite for all $i \in \omega$) induces an endomorphism $\hat{t}$ of $\mathfrak{C}(\omega)$ via $\hat{t}(s) = \{i \mid t(i) \in s\}$. It is easy to check that t is one-to-one (onto) iff $\hat{t}$ is onto (one-to-one). An appropriate choice of t finishes the proof. $\square$

COROLLARY 4. *There are no countable hopfian Boolean algebras.*

PROOF. Let $\mathfrak{A}$ be countably infinite. By Theorem 3, we may assume that $\mathfrak{A}$ has only finitely many atoms with sum s. So $\mathfrak{A} \cong \mathfrak{A} \restriction s \times \mathfrak{A} \restriction -s$ where $\mathfrak{A} \restriction -s$ is isomorphic to the countable atomless BA $\mathfrak{F}$. Let $\{i_n: n \in \omega\}$ be the independent generators of $\mathfrak{F}$. Then h_1 defined by $h_1(i_0) = h(i_1) = i_0$ and $h_1(i_{n+1}) = i_n$ for $n > 0$ can be extended to an endomorphism h of $\mathfrak{F}$ which is onto but not one-to-one. When "attached" to the identity on $\mathfrak{A} \restriction s$, h clearly demonstrates that $\mathfrak{A}$ is not hopfian. $\square$

There is a nice translation of some of the above into topological terms, as was pointed out by the referee. In this paragraph, consider only BA's with infinitely many atoms. For such an algebra $\mathfrak{A}$, let X be its Stone space. The conclusion of Lemma 2 is easily seen to be equivalent to the following property denoted by (*): There is a countably infinite subset D of X of isolated points such that the closure $\overline{D}$ of D in X has $|\overline{D} \sim D| = 1$. Every superatomic BA (every subalgebra is atomic) and every BA with an ordered base satisfies (*). In particular, every countable BA satisfies (*). Obviously, if $\mathfrak{A}$ satisfies (*), then $\mathfrak{A}$ is not hopfian. For if $D = \{d_n: n \in \omega\}$, define $f: X \to X$ by $f(d_n) = d_{n+1}$ and $f(x) = x$ for $x \in X \sim D$. Then f is continuous, one-to-one, and not onto.

The results above also hold for BA's which might be called dual hopfian; i.e. every one-to-one endomorphism is onto. The proofs require little modification.

In Theorem 3, the requirement that $\mathfrak{A}$ have infinitely many atoms is necessary when $\mathfrak{A}$ is uncountable, since there are atomless (in fact, rigid) hopfian BA's in every uncountable cardinality. (See Loats and Rubin [1].) Also, note that Corollary 4 and the results in [1] yield a complete description of the cardinalities in which atomless hopfian BA's can occur.

We now turn to the construction of an atomic hopfian BA of power $2^{\aleph_0}$.

LEMMA 5. *Let* $\{d_n | n \in \omega\}$ *be a family of pairwise disjoint nonempty subsets of* ω *satisfying*

(∇): *If* $|d_n| = 1$, *for all* $n \in \omega$, *then there is an infinite set* $S \subseteq \omega$ *such that* $n \in S$ *iff* $d_n \neq \{n\}$. *Then there is an infinite* $M \subseteq \omega$ *such that*

$$M \cap \bigcup_{n \in M} d_n = \varnothing.$$

PROOF. *Case* 1. $|d_n| = 1$ for all $n \in \omega$. Obvious.

Case 2. There is n_0 with $|d_{n_0}| > 1$. By induction, choose $n_1 \in d_{n_0} \sim \{n_0\}$, and for $k \geqslant 1$, choose $n_{k+1} \in d_{n_k}$ such that $n_{k+1} \neq n_1$ if $n_k = n_0$. By straightforward induction on j one shows that $1 \leqslant i < j$ implies $n_i \neq n_j$. Let $M = \{n_{2i} | 0 \neq i \in \omega\}$. $\square$

THEOREM 6. *There is an atomic hopfian BA* $\mathfrak{A} \subseteq \mathcal{P}(\omega)$ *of power* $2^{\aleph_0}$ *such that* $|\mathrm{Aut}\, \mathfrak{A}| = \aleph_0$.

PROOF. For each $\alpha < 2^{\aleph_0}$, let $\langle d_n^\alpha \colon n \in \omega\rangle$ be a sequence of pairwise disjoint nonempty subsets of ω satisfying (∇) of Lemma 5. Furthermore, assume that every infinite partition of ω and each of its permutations appear $2^{\aleph_0}$ times in the sequence $\langle\langle d_n^\alpha \colon n \in \omega\rangle \colon \alpha < 2^{\aleph_0}\rangle$.

Let $\mathfrak{A}_0 = \mathfrak{C}(\omega)$ and set $B_0 = \varnothing$. We shall define $\mathfrak{A}_\alpha$, B_α for each $\alpha < 2^{\aleph_0}$ so that

$$|A_\alpha|, |B_\alpha| \leqslant |\alpha| + \aleph_0$$

and

$$A_\alpha \cap B_\alpha = \varnothing.$$

For a limit ordinal $\lambda < 2^{\aleph_0}$, set

$$\mathfrak{A}_\lambda = \bigcup_{\alpha < \lambda} \mathfrak{A}_\alpha \quad \text{and} \quad B_\lambda = \bigcup_{\alpha < \lambda} B_\alpha.$$

Let $\alpha < 2^{\aleph_0}$. In order to define $\mathfrak{A}_{\alpha+1}$ and $B_{\alpha+1}$, consider $\langle d_n^\alpha \colon n \in \omega\rangle$. For convenience, we suppress the index α. If $\langle d_n \colon n \in \omega\rangle \nsubseteq A_\alpha$, then set $\mathfrak{A}_{\alpha+1} = \mathfrak{A}_\alpha$ and $B_{\alpha+1} = B_\alpha$. So assume that $\{d_n \colon n \in \omega\} \subseteq A_\alpha$. By Lemma 5, there is an infinite $M \subseteq \omega$ such that $M \cap \bigcup_{n \in M} d_n = \varnothing$. Set

$$D_\alpha = \left\{ \bigcup_{n \in S} d_n | S \subseteq M \right\}.$$

For each $T \in D_\alpha$, set $f_\alpha(T) = \{n | d_n \subseteq T\}$. Notice $T \cap f_\alpha(T) = \varnothing$ for all $T \in D_\alpha$. We shall show:

(1) There is $J \in D_\alpha \sim A_\alpha$ such that

$$\mathrm{Sg}(A_\alpha \cup \{J\}) \cap (B_\alpha \cup \{f_\alpha(J)\}) = \varnothing.$$

For if (1) holds, set

$$\mathfrak{A}_{\alpha+1} = \mathrm{Sg}(A_\alpha \cup \{J\}),$$
$$B_{\alpha+1} = B_\alpha \cup \{f_\alpha(J)\}$$

and proceed by induction. But if not, we have:

(2) For every $J, J \in D_\alpha \sim A_\alpha$, there exist $C, D \in A_\alpha$ such that

$$(C \cap J) \cup (D \sim J) \in B_\alpha \cup \{f_\alpha(J)\}.$$

Let $K_0 \subseteq D_\alpha$ be of power $2^{\aleph_0}$ such that for $J_0, J_1 \in K_0, f_\alpha(J_0) \cap f_\alpha(J_1)$ is a finite subset of ω. By (2), we have

$$K_0 \sim A_\alpha = \bigcup_{C,D \in A_\alpha} \{J \in K_0 \sim A_\alpha | (C \cap J) \cup (D \sim J) \in B_\alpha \cup \{f_\alpha(J)\}\}.$$

Since $|A_\alpha|, |B_\alpha| < 2^{\aleph_0}$ and $|K_0 \sim A_\alpha| = 2^{\aleph_0}$, there exist $C, D \in A_\alpha$ such that for

$$K_1 = \{J \in K_0 \sim A_\alpha | (C \cap J) \cup (D \sim J) \in B_\alpha \cup \{f_\alpha(J)\}\},$$

we have $|K_1| > |A_\alpha| + |B_\alpha|$. So

(3) There is at most one $J \in K_1$ such that $(C \cap J) \cup (D \sim J) = f_\alpha(J)$.

For if not, let J_0, J_1 be two such J's. Since $J_i \cap M = \varnothing$ and $f_\alpha(J) \subseteq M$, it is easy to see that $f_\alpha(J_i) = D \sim J_i$. In fact, one can even show for $i = 0,1$, $f_\alpha(J_i) = D \cap M$. But since J_0 and J_1 are distinct, so are $f_\alpha(J_0)$ and $f_\alpha(J_1)$, a contradiction.

Let

$$K_2 = \bigcup_{E \in B_\alpha} \{J \in K_1 | (C \cap J) \cup (D \sim J) = E\}.$$

By (3), $|K_2| > |B_\alpha|$, so there is $E \in B_\alpha$ such that for

$$K_3 = \{J \in K_2 | (C \cap J) \cup (D \sim J) = E\},$$

we have $|K_3| > |B_\alpha| + \aleph_0$. In particular, we may choose distinct $J,H \in K_3$. Now

$$E = [C \cap J \cap H] \cup [C \cap D] \cup [D \sim (H \cap J)].$$

Since $J \cap H \in A_\alpha$ as are C,D, we have $E \in A_\alpha \cap B_\alpha$, a contradiction. Therefore, (1) holds. Next,

(4) $\mathfrak{A} = \mathfrak{A}_{2^{\aleph_0}}$ is the desired BA. For, let $h \in \operatorname{End} \mathfrak{A}$ such that

(a) h is onto, and

(b) h is not an automorphism of $\mathfrak{A}$ induced by a permutation of At $\mathfrak{A}$ with finite support.

By induction, we choose a family $\langle d_n : n \in \omega \rangle$ of pairwise disjoint non-empty subsets of ω so that $h(d_n) = \{n\}$ for all n. In case h is not one-to-one, it is clear that we may insure that at least one d_n has two or more elements. It remains to check that $\langle d_n : n \in \omega \rangle$ satisfies (∇): By our choice above, if $|d_n| = 1$ for all n, then h is an automorphism. By condition (b), there must exist an infinite $S \subseteq \omega$ such that $h \in S$ iff $d_n \neq \{n\}$, as desired.

Since $cf(2^{\aleph_0}) > \aleph_0$, there is $\alpha < 2^{\aleph_0}$ such that

$$\langle d_n : n \in \omega \rangle = \langle d_n^\alpha : n \in \omega \rangle \subseteq A_\alpha.$$

At stage α (see (1)), we put a $J \in D_\alpha$ into $A_{\alpha+1}$ so that $f_\alpha(J) \notin A$. But

$f_\alpha(J) = h(J)$. For if $d_n \subseteq J$, then $\{n\} = h(d_n) \subseteq h(J)$. If $d_n \nsubseteq J$, then $d_n \cap J = \varnothing$. So $\{n\} \cap h(J) = \varnothing$. Thus $f_\alpha(J) = h(J) \notin A$, a final contradiction. $\square$

Problem 7. Are there any atomic hopfian BA's with uncountably many atoms?

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ADDED IN PROOF. I have recently noticed that the proof of Theorem 6 may be modified to obtain an atomic Boolean algebra of power $2^{\aleph_0}$ which is dual hopfian. In fact, the two constructions can be merged to obtain an algebra which is both hopfian and dual hopfian.

Judy Roitman and myself have recently shown that it is consistent with ZFC + not CH that the answer to Problem 7 is yes.

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