ISOMETRIES ON $L^p$ SPACES AND COPIES OF $L^p$ SHIFTS

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Abstract. Necessary and sufficient conditions are given on an isometry $V$ in an $L^p$ space so that there exists an invariant subspace $M$ such that $V$ restricted to $M$ is isometrically equivalent to the unilateral shift on $L^p$.

1. Introduction. The unilateral shift on $L^p$ is probably the best known and studied of the nonunitary isometries on $L^p$ spaces. Even on $L^p$, isometries $V$ such that $\cap V^n = \{0\}$ need not be isometric to the unilateral shift on $L^p$. This note will characterize when an isometry on $L^p$ contains (in the appropriate sense) a copy of the unilateral shift on $L^p$. The question of in what sense isometries in more general Banach spaces look like shifts is discussed in [1].

In what follows $L^p$ will be $L^p(X, \Sigma, \mu)$ where $(X, \Sigma, \mu)$ is a $\sigma$-finite measure space, $1 < p < \infty, p \neq 2$. $L^p$ is, of course, $L^p$ with $X$ the nonnegative integers, $\Sigma$ all subsets of $X$, and $\mu$ counting measure. The unilateral shift $S$ on $L^p$ is given by $S(\alpha_0, \alpha_1, \ldots) = (0, \alpha_0, \alpha_1, \ldots)$. $V$ will always denote an isometry on $L^p$. From [2] there exists a regular set isomorphism $T: \Sigma \rightarrow \Sigma$ so that

$$(Vf)(x) = h(x)(T(f))(x). \tag{1}$$

Formula (1) should be interpreted in the following sense.

If $C \in \Sigma$, let $1_C$ denote its characteristic function. Then $T(1_C) = 1_{T(C)}$ and $T$ is extended to simple functions by linearity. The map $V$ as expressed in (1) is an isometry on $L^p$ simple functions and hence extends to all of $L^p$. Without loss of generality, we may assume $\mu(\{x|h(x) = 0\}) = 0$. Note that $T(1_C)$ need not be in $L^p$ even if $1_C$ is. The only requirement is that $\|h1_{T(C)}\| = (\mu(C))^{1/p}$, so that $hT(1_C)$ is in $L^p$. It should also be pointed out that the characterization of $h$ in [2] is vague. It is not necessary that $|h| = d\mu^*/d\mu$, $\mu^* = \mu \circ T^{-1}$. It is sufficient that

$$\int_{T(A)} |h|^p d\mu = \int_{T(A)} \frac{d\mu^*}{d\mu} d\mu, \quad A \in \Sigma.$$ 

If $T(\Sigma)$ is a properly smaller Boolean algebra, then one need not have $|h|^p = d\mu^*/d\mu$.

All statements about sets and functions are modulo sets of $\mu$-measure zero. For $f \in L^p$, supp$(f)$ denotes its support.
2. Main results. The main result of this note is the following characterization of when an isometry on $L^p$ 'contains' a copy of $S$ on $l^p$.

**Theorem 1.** Suppose that $V: L^p \to L^p$ is an isometry. Then the following statements (I)–(III) are equivalent.

(I) There exists a subspace $M \subseteq L^p$ such that:

(i) $VM \subseteq M$,

(ii) $M$ is isometric to $l^p$,

(iii) $US = VU$ where $U$ is the isometric map of (ii), $U: l^p \to M$;

(II) There exists a set $A \in \Sigma$ such that $T(A) \subseteq A$, $0 < \mu(A \setminus T(A)) < \infty$;

(III) There exists an $f \in L^p$ such that $\text{supp}(Vf) \subseteq \text{supp}(f)$ and $0 < \mu(\text{supp}(f) \setminus \text{supp}(Vf)) < \infty$.

**Proof.** We shall prove (I) $\Rightarrow$ (II) first. Assume that (I) holds. Let $e_i = 1_{(i)} \in l^p$. Let $A_i = \text{supp}(U1_{(i)})$. Note that $A_i \cap A_j = \emptyset$ if $i \neq j$ since

$$\|Ue_i \pm Ue_j\|^p = \|e_i \pm e_j\|^p = \|e_i\|^p + \|e_j\|^p,$$

implies $Ue_i Ue_j = 0$ almost everywhere $\mu$ [2]. Note also that from (iii) $T(A_i) \subseteq A_{i+1}$. If $\mu(A_i) < \infty$, let $A = \cup A_i$ and (II) holds. If $A_1$ is not of finite measure, let $E_i$ be a subset of $A_i$ such that $\mu(E_i) < \infty$. ($E_i$ exists since $A_1$ is the support of an $L^p$ function.) Let $A = E_1 \cup \cup_{n=1}^{\infty} T^n(E_i)$. Then (II) holds. To see that (II) $\Rightarrow$ (I), assume that (II) holds. First we need to show that:

If $f \in L^p$ and $\text{supp}(f) \subseteq D$, then $\text{supp}(V^n f) \subseteq T^n(D)$ for $n > 1$. (2)

To prove (2), it suffices to prove the $n = 1$ case. Then $n > 1$ follows by induction. If $g \in L^p$ is a simple function, so that

$$g = \sum_{i=1}^{n} a_i \delta_{E_i}, \quad E_i \cap E_j = \emptyset \text{ if } i \neq j, \quad a_i \neq 0,$$

then

$$Vg = \sum_{i=1}^{n} a_i V1_{E_i} = \sum_{i=1}^{n} a_i h(x) 1_{T(E_i)}.$$

(Note that $T(E_i) \cap T(E_j) = \emptyset$.) Thus $\text{supp}(Vg) = \cup T(E_i) \subseteq T(D)$ if $\cup E_i \subseteq D$. Thus (2) holds for simple functions. Now if $f \in L^p$, chose simple functions $f_n$ so that $f_n \to f$ in $L^p$ norm and $\text{supp}(f_n) \subseteq \text{supp}(f) \subseteq D$. But $Vf_n \to Vf$. Take a subsequence so that $Vf_n \to Vf$ almost everywhere. But $\text{supp}(Vf_n) \subseteq T(D)$. Thus $\text{supp}(Vf) \subseteq T(D)$ and (2) follows. By a similar argument using simple functions it is straightforward to show:

If $f, g \in L^p$ and $\text{supp}(g) = \text{supp}(f)$, then $\text{supp}(Vg) = \text{supp}(Vf)$. (3)

Now to prove (I). By assumption there is an $A \in \Sigma$ such that $T(A) \subseteq A$, $0 < \mu(A \setminus T(A)) < \infty$. Let $B = A \setminus T(A)$. Since $T(A) \subseteq A$, $T^n(A) \subseteq T(A)$ for $n > 1$. Thus $B \cap T^n(B) = \emptyset$. Since $T$ is regular, this implies that $T^k(B) \cap T^l(B) = \emptyset$ for $k \neq l$. Let $g = ah1_B$ where $a = \|h1_B\|^{-1}$. From (2), (3), it follows that $\text{supp}(V^n g) \cap \text{supp}(V^m f) = \emptyset$ for $m \neq n$. Let $g_n = V^n g$. For $\sum a_n e_n \in l^p$, define $U: l^p \to L^p$ by $U(\sum a_n e_n) = \sum a_n g_n$. Let $M = U1^p$. Since $\text{supp}(g_n) \cap \text{supp}(g_m) = \emptyset$ if $n \neq m$ and $\|g_n\| = 1$, $U$ is an isometry so that
(ii) holds. That (i) and (ii) hold is clear and (I) follows. That (III) is equivalent to (I) and (II) is now clear. □

3. Comments. A natural question is whether (I) always holds; that is, whether every isometry on an $L^p$ space contains a copy of the unilateral shift on $l^p$. If $V$ is unitary, the answer is, in general, no. The identity is an example. What about nonunitary isometries? Do they always satisfy (I)? We shall show that the answer is no for general $L^p$ and affirmative for nonunitary isometries on $l^p$.

**Example.** Let $X = [0, 2]$, and $\mu$ be Lebesgue measure. For $A \subseteq \Sigma$, define $T(A) = \frac{1}{2} A \cup \frac{1}{2} A + 1$. (Here $\frac{1}{2} A = \{\frac{1}{2} a : a \in A\}$, $\frac{1}{2} A + 1 = \{\frac{1}{2} a + 1 : a \in A\}$.) Clearly $T$ is a measure-preserving transformation of $\Sigma$ into itself. Let $h$ be identically 1 and define $V$ by (1). Then $V$ is an isometry of $L^p$ into itself. Let $g = 1_{[0,1]} - 1_{[1,2]}$. Now $\int_{T(A)} g \, d\mu = 0$ for any set $A \subseteq \Sigma$. Hence $\int (Vf)g \, d\mu = 0$ for all $f \in L^p$. Since $g \in (L^p)^*$, we have $V$ is not onto and hence $V$ is not unitary. But $V$ cannot satisfy (II). For suppose there existed $A \subseteq \Sigma$ such that $T(A) \subset A$, $0 < \mu(A \cap T(A)) < \infty$. Let $B = A \setminus T(A)$. Then $2 > \mu(\bigcup T^n(B)) = \sum \mu(T^n(B)) = \sum \mu(B)$ which is impossible. Note that this $V$ also satisfies $\text{R}(V^n) = \{0\}$.

This example is a special case of the more general fact that:

**Proposition.** If $V : L^p \to L^p$ is an isometry and satisfies (II), then either $\mu(X)$ is not finite or $T$ is not measure-preserving.

We conclude by showing that:

**Theorem 2.** If $V$ is a nonunitary isometry on $l^p$, then it satisfies (I).

**Proof.** Represent $V$ as in (1), and assume that $V$ is not unitary. Without loss of generality, we may assume $i \not= T^m(i)$ for all $i$ and $m > 0$. (These unitary summands may be discarded, if present.) Pick an $i$. If $i \not\in T^m(i)$ for all $m > 0$, then (II) holds. Thus we may assume that $i \in T^m(i)$ for some $m$. Take $m$ to be the least such $m$. If $m = 1$, let $B = T(i) \setminus \{i\}$. Then $i \not\in T(B)$ and by induction $i \not\in T^m(B)$ for all $m$. Hence $B \cap T^m(B) = \emptyset$ for all $m > 0$ and (II) holds for $A = \bigcup T^m(C)$, $C$ any finite subset of $B$. Thus we may assume $m > 1$. Now there exist distinct $i_1, \ldots, i_{m-1}$ such that $i \in T(i_{k-1})$, $i_0 = i$, $i \in T(i_{m-1})$. The distinctness of the $i_j$ follows from the fact that $m$ is minimal. Let $I = \{i, i_1, \ldots, i_{m-1}\}$, and $B = T(I) \setminus I$. Note that by assumption $B \neq \emptyset$. Now $I \cap C = \emptyset$ implies $I \cap T(C) = \emptyset$ for any set $C$. But $B \cap I = \emptyset$. Hence $T^n(B) \cap I = \emptyset$ for all $n > 0$. But then $B \cap T^n(B) = \emptyset$ for all $n > 0$ and again (II) holds. □

**Bibliography**


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