PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 77, Number 2, November 1979

AN ABEL-TAUBER THEOREM ON CONVOLUTIONS WITH THE MÖBIUS FUNCTION

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ABSTRACT. Suppose n: $R^+ \to R^+$ and $n(x)/x$ is integrable on $(0, \infty)$. For $s > 0$ we define

$$\tilde{n}(s) = s\int_0^\infty \frac{e^{-us}}{1 - e^{-us}} n(u)\, du.$$

In this paper an Abel-Tauber theorem is proved concerning this transform. Moreover the relation between $\tilde{n}(s)$ and $\sum_{m \leqslant s} n(s/m)/m$ is studied.

Introduction. Following earlier work by Landau [**10**] and others, Ingham proved (in [**7**]) the following theorem:

THEOREM A. *Suppose that*

(i) $f(x)$ *is positive and nondecreasing for* $x \geqslant 1$,

(ii) $F(x) \equiv \sum_{n \leqslant x} f(x/n) = ax\log x + bx + o(x)$ $(x \to \infty)$, *where* a *and* b *are constants. Then*

(a) $f(x) \sim ax (x \to \infty)$,

(b) $\int_1^\infty (f(x) - ax)/x^2\, dx = b - a\gamma$ *where* γ *is Euler's constant.*

Moreover, for any function $f(x)$, bounded and integrable over every finite interval $(1, X)$ hypothesis (ii) implies conclusion (b). More recently Jukes [**8**] extended results of Segal (see [**13**], [**14**]) and proved the following theorem which can be considered as a generalization of Theorem A. For a proof of this theorem see also [**15**].

THEOREM B. *Let* $f(x)$ *be bounded and integrable on every finite subinterval of* $[1, \infty)$ *and suppose that*

$$\sum_{n \leqslant x} f\left(\frac{x}{n}\right) = xg(x) + o(x^2 g'(x)) \qquad (x \to \infty)$$

where $g(x) \in C^2[1, \infty)$ *is positive and satisfies*

(i) $g'(x) > 0$ *for all* $x \in [1, \infty)$,

(ii) *for some real* r, $xg'(x)(\log x)^{-r}$ *is nonincreasing for* $x > x_0$,

(iii) *for some real* s, $xg'(x)(\log x)^s \equiv q(x)$ *is nondecreasing for* $x > x_1$ *and* $q(x) \to \infty$ $(x \to \infty)$,

Then $\int_1^x f(t)/t^2\, dt = g(x) - \gamma x g'(x) + o(xg'(x))$ $(x \to \infty)$.

Received by the editors June 15, 1978.

AMS (MOS) subject classifications (1970). Primary 40E05; Secondary 26A12.

Key words and phrases. Abel-Tauber theorems, regular variation.

If we suppose $f(x)$ to be positive and nondecreasing we also have $f(x) \sim x^2 g'(x)$ $(x \to \infty)$.

In this note we consider the situation in Theorem B where $g(x)$ is allowed to grow faster than in (ii).

We use the concept of regular variation as introduced by Karamata to formulate natural conditions on $g(x)$ and prove a theorem similar to Theorem B. Moreover, an application is given which gives a second-order condition in a relation considered by Parameswaran in [**11**]. See our Corollary 1 and the remark after Theorem 4. Parameswaran proves the following theorem [**11**, Theorem II].

THEOREM C. *If $\int_0^R n(u)/u\, du$ exists in the Lebesgue sense for every positive R*

$$f(s) \equiv \exp\left\{ s \int_0^\infty \frac{e^{-su}}{1 - e^{-su}}\, n(u)\, du \right\}$$

for all positive s and $n(u)$ nondecreasing, then the relation $\log f(s) \sim \int_a^{1/s} L(u)/u\, du$ as $s \to 0^+$ implies the relation $n(u) \sim L(u)$ as $u \to \infty$, provided $L(x)$ is a slowly oscillating function defined for $x \geqslant a$ such that $L(x) \sim K \exp\{\int_a^x \delta(u)/u\, du\}$ as $x \to \infty$ where δ is a nonincreasing function and K is a positive real number.

We use our Theorem 3 to get a theorem similar to Theorem C.

RESULTS. A real valued function q is said to be regularly varying at infinity if it is positive and measurable on $[A, \infty)$ for some $A > 0$ and if for each $\lambda > 0$, $\lim_{x\to\infty} q(\lambda x)/q(x) = \lambda^\rho$ for some $\rho \in (-\infty, \infty)$. ρ is called the exponent of regular variation. Functions q which satisfy the limit relation with $\rho = 0$ are called slowly varying (slowly oscillating).

For an extensive treatment of regular variation the reader is referred to Seneta [**16**].

Now we define the subclass of slowly varying functions we want to consider.

DEFINITION. The nondecreasing function q: $R^+ \to R$ belongs to the class Π if there exists a slowly varying function L such that $q(x) = \int_1^x L(t)/t\, dt + L(x)$.

It can be shown that the class Π consists of all nondecreasing functions q: $R^+ \to R$ for which there exist functions a: $R^+ \to R^+$ and b: $R^+ \to R$ such that for all positive x

$$\lim_{t\to\infty} \frac{q(tx) - b(t)}{a(t)} = \log x.$$

If the last relation holds, then it is true with $b(t) = q(t)$ and $a(t) = q(te) - q(t)$. The function $a(t)$ is of course determined up to asymptotic equivalence and is called the auxiliary function. As a consequence of these facts we mention the following result: if $q(x) \in \Pi$ with auxiliary function $L(x)$, $q_1(x)$ is nondecreasing and $(q(x) - q_1(x))/L(x) \to C$ $(x \to \infty)$, where $C \in R$ is a

constant, then $q_1(x) \in \Pi$ with auxiliary function $L(x)$. It is easy to see that if $L(x)$ slowly varying, then $\int_1^x L(t)/t\,dt$ is an element of Π with auxiliary function $L(x)$.

The class Π is a subclass of the slowly varying functions at infinity. Moreover, for each function $q \in \Pi$ it is possible to find a function L_* such that $q(x) = \int_1^x L_*(t)/t\,dt + o(L(x))$ $(x \to \infty)$ where $L_* \sim L$ is slowly varying.

For the proofs of these statements and further properties see [3], [4] and [5].

We start with two Abelian results.

THEOREM 1. *Suppose n: $R^+ \to R^+$ is slowly varying, $n(x)/x$ is (Lebesgue) integrable on finite subintervals of* $(0, \infty)$. *Then $\tilde{n}(1/s) \in \Pi$ with auxiliary function $n(s)$, where $\tilde{n}$ is defined by*

$$\tilde{n}(s) = s\int_0^\infty \frac{e^{-us}}{1 - e^{-us}}\, n(u)\, du.$$

Moreover,

$$\frac{\tilde{n}\left(\frac{1}{s}\right) - \int_0^s \frac{n(t)}{t}\, dt}{n(s)} \to 0 \qquad (s \to \infty).$$

PROOF.

$$\frac{\int_0^s \frac{n(t)}{t}\, dt - \tilde{n}\left(\frac{1}{s}\right)}{n(s)}$$

$$= \int_0^1 \left(\frac{1}{u} - \frac{e^{-u}}{1 - e^{-u}}\right) \frac{n(su)}{n(s)}\, du - \int_1^\infty \frac{e^{-u}}{1 - e^{-u}}\, \frac{n(su)}{n(s)}\, du.$$

We have $\int_0^1 n(su)/n(s)\,du = \int_0^s n(v)dv/sn(s) \to 1$ $(s \to \infty)$ by Karamata's theorem on regularly varying functions and since $1/u - e^{-u}/(1 - e^{-u})$ is bounded on $(0, 1)$ we can apply Pratt's lemma (see [12]).

For the second part we have $(su)^{-\varepsilon}n(su)/s^{-\varepsilon}n(s) \to u^{-\varepsilon}$ $(s \to \infty)$ uniform on $(1, \infty)$ by Corollary 1.2.1.4 of [3]. So we have

$$\frac{\int_0^s \frac{n(t)}{t}\, dt - \tilde{n}\left(\frac{1}{s}\right)}{n(s)} \to \int_0^1 \left(\frac{1}{u} - \frac{e^{-u}}{1 - e^{-u}}\right) du - \int_1^\infty \frac{e^{-u}}{1 - e^{-u}}\, du \qquad (s \to \infty).$$

The right side is zero, as is shown by elementary integration. This implies that $\tilde{n}(1/s) \in \Pi$ with auxiliary function $n(s)$, since $\int_0^s n(t)/t\,dt \in \Pi$ with auxiliary function $n(s)$ and $\tilde{n}(1/s)$ is nondecreasing.

REMARKS. 1. The case $n \in RV_\alpha^{(\infty)}$ $(\alpha > 0)$ (regularly varying at infinity with exponent α) gives analogously $\tilde{n}(1/s)/n(s) \to \zeta(\alpha + 1)\Gamma(\alpha + 1)$ $(s \to \infty)$. See [9, Theorem 1].

2. The statement of the theorem implies $\tilde{n}(1/s) \sim \int_0^s n(t)/t\,dt$ since $n(s) = o(\int_0^s n(t)/t\,dt)$ by Karamata's theorem. See Theorem I of [**11**].

THEOREM 2. *If we define* $\Psi(s) = \sum_{m \leqslant s} n(s/m)/m$ *where* n *satisfies the conditions of Theorem* 1 *and* $xn(x)$ *is of bounded variation on intervals of the form* $(1, x_0)$, *then*

$$\left(\Psi(s) - \int_1^s \frac{n(t)}{t}\,dt\right) \Big/ n(s) \to \gamma \qquad (s \to \infty).$$

PROOF. We give the proof by an Euler-Maclaurin kind of argument

$$\int_{\nu-1}^{\nu} \frac{x}{t}\, n\left(\frac{x}{t}\right) dt = xn\left(\frac{x}{\nu}\right) - xn\left(\frac{x}{\nu-1}\right) - \int_{\nu-1}^{\nu} t d\frac{x}{t}\, n\left(\frac{x}{t}\right)$$
$$= \frac{x}{\nu}\, n\left(\frac{x}{\nu}\right) - \int_{\nu-1}^{\nu} \{t\} d\frac{x}{t}\, n\left(\frac{x}{t}\right)$$

where we use the notation $\{t\} = t - [t]$.

Summing over ν gives

$$\sum_{1 \leqslant \nu \leqslant x} \frac{1}{\nu}\, n\left(\frac{x}{\nu}\right) - \int_1^x \frac{1}{t} n\left(\frac{x}{t}\right) dt$$
$$= n(x) - \int_{[x]}^{x} \frac{1}{t}\, n\left(\frac{x}{t}\right) dt + \int_1^{[x]} \{t\} d\frac{1}{t}\, n\left(\frac{x}{t}\right).$$

Now with $M = \sup_{x \in (1,2)} n(x)$, we have

$$\left|\int_{[x]}^{x} \frac{1}{t}\, n\left(\frac{x}{t}\right) dt \Big/ n(x)\right| \leqslant \left(M \log \frac{x}{[x]}\right) \Big/ n(x) \leqslant \frac{M}{n(x)(x-1)} \to 0 \qquad (x \to \infty)$$

since $(x-1)n(x)$ is 1-varying at infinity. Furthermore

$$\int_1^{[x]} \{t\} d\frac{1}{t}\, n\left(\frac{x}{t}\right) = -\frac{1}{x} \int_1^{[x]} \frac{x}{t}\, n\left(\frac{x}{t}\right) d\{t\}$$
$$\sim -n(x) \int_1^{[x]} \frac{1}{t}\, d\{t\} \sim (\gamma - 1) n(x)$$

since

$$\left(\frac{x}{t} n\left(\frac{x}{t}\right)\right) \Big/ xn(x) \to t^{-1}$$

uniform for $t \in (1, \infty)$ (see [**3**, Corollary 1.2.1.4]) and $\int_1^\infty \{t\}/t^2\,dt = 1 - \gamma$ (see e.g. [**6**]). This proves Theorem 2.

REMARKS. 1. The theorem implies $(\Psi(tx) - \Psi(t))/(\Psi(te) - \Psi(t)) \to \log x$ $(t \to \infty)$ where $x > 0$ is fixed. If Ψ is nondecreasing we have $\Psi \in \Pi$.

2. With the additional supposition that Ψ is nondecreasing, a simpler proof is possible:

$$\tilde{n}(s) - s\int_0^1 \frac{e^{-us}}{1 - e^{-us}}\, n(u)\,du = s\int_1^\infty \sum_{k=1}^{\infty} e^{-kus} n(u)\,du = \int_0^\infty e^{-us}\,d\Psi(u).$$

Since

$$\frac{1}{s}\int_0^1 \frac{e^{-u/s}}{1-e^{-u/s}}\, n(u)\, du - \int_0^1 \frac{n(u)}{u}\, du = o(n(s)) \qquad (s \to \infty)$$

as in Theorem 1 we have

$$\left(\tilde{n}\left(\frac{1}{s}\right) - \int_0^1 \frac{n(u)}{u}\, du - \Psi(s)\right) \Big/ n(s) \to -\gamma \qquad (s \to \infty)$$

(see [**5**, Theorem 1]). Combining this with Theorem 1 gives the desired result.

3. The case $n \in RV_\alpha^{(\infty)}$ $(\alpha > 0)$ in Remark 2 gives $\Psi(s)/(\int_0^s n(t)/t\, dt) \to \alpha\zeta(\alpha + 1)$ $(s \to \infty)$.

4. The statement of the theorem implies $\Psi(s) \sim \int_0^s n(t)/t\, dt$ since $n(s) = o(\int_0^s n(t)/t\, dt)$ $(s \to \infty)$.

As a consequence of Remark 2 we mention the following.

COROLLARY 1. *Suppose n: $R^+ \to R^+$ satisfies the conditions $\int_0^R n(u)/u\, du < \infty$ for $R > 0$ and $\sum_{m \leq s} n(s/m)/m$ is nondecreasing for $s > 0$. The assertions $\sum_{m \leq s} n(s/m)/m \in \Pi$ with auxiliary function $L(s)$ and $\tilde{n}(1/s) \in \Pi$ with auxiliary function $L(s)$ are equivalent. Both imply*

$$\left(\sum_{m \leq s} \frac{1}{m}\, n\left(\frac{s}{m}\right) - \tilde{n}(1/s) + \int_0^1 \frac{n(t)}{t}\, dt\right) \Big/ L(s) \to \gamma \qquad (s \to \infty).$$

For the Tauberian counterpart of Theorems 1 and 2 we need three lemmas.

LEMMA 1. *If*

(i) *$L(x)$ is nondecreasing for $x > 0$, slowly varying, $L(x) \to \infty$ $(x \to \infty)$,*

(ii) *$L(x)/L(x-1) \leq 1 + x^{-\alpha}$ for some $\alpha > 0$, $x \geq x_0 = x_0(\alpha)$, then $\sum_{m \leq x} (\mu(m)/m) L(x/m) = o(L(x))$ $(x \to \infty)$.*

PROOF. We define $a_n = L(n) - L(n-1)$ for $n \geq 2$, $a_1 = L(1)$. Then

$$\sum_{m \leq x} \frac{\mu(m)}{m}\, L\left(\left[\frac{x}{m}\right]\right) = \sum_{m \leq x} \frac{\mu(m)}{m} \sum_{\nu \leq x/m} a_\nu = \sum_{m \leq x} a_m N\left(\frac{x}{m}\right)$$

where $N(x) = \sum_{m \leq x} \mu(m)/m$. Since $|N(x)| < \varepsilon$ for $x \geq x_\varepsilon$ (see [**10**]) we have

$$\left|\sum_{m \leq x/x_\varepsilon} a_m N\left(\frac{x}{m}\right)\right| < \varepsilon \sum_{m \leq x/x_\varepsilon} a_m = \varepsilon L\left(\left[\frac{x}{x_\varepsilon}\right]\right) \leq \varepsilon(1+\varepsilon) L(x) \qquad (x > x_\varepsilon)$$

and

$$\left|\sum_{x/x_\varepsilon < m \leq x} a_m N\left(\frac{x}{m}\right)\right| \leq c \sum_{x/x_\varepsilon < m \leq x} a_m \leq c\left\{L(x) - L\left(\frac{x}{x_\varepsilon}\right)\right\} = o(L(x)) \qquad (x \to \infty).$$

For $x \geq x_0(\alpha)$ we have by (ii), $L(x) - L([x]) \leq L(x) - L(x-1) \leq L(x)/x^\alpha$ since L is nondecreasing. Hence,

$$\left|\sum_{1\leqslant m\leqslant x/x_0}\frac{\mu(m)}{m}\left\{L\left(\frac{x}{m}\right)-L\left(\left[\frac{x}{m}\right]\right)\right\}\right|\leqslant\sum_{m\leqslant x/x_0}\frac{1}{m}\frac{L(x/m)}{x^\alpha/m^\alpha}$$
$$\leqslant x^{-\alpha}\left\{\int_1^{x/x_0}\frac{1}{u^{1-\alpha}}L\left(\frac{x}{u}\right)du+L(x)\right\}$$
$$=\int_{x_0}^x\frac{L(v)}{v^{1+\alpha}}\,dv+o(L(x))=o(L(x)),$$

if we choose $0<\alpha<1$.

We estimate the last sum as follows.

$$\left|\sum_{x/x_0<m\leqslant x}\frac{\mu(m)}{m}\left\{L\left(\frac{x}{m}\right)-L\left(\left[\frac{x}{m}\right]\right)\right\}\right|$$
$$\leqslant 2\sum_{x/x_0<m\leqslant x}\frac{1}{m}L\left(\frac{x}{m}\right)\leqslant 2L(x_0)\sum_{x/x_0<m\leqslant x}\frac{1}{m}$$
$$=O(1)=o(L(x))\qquad(x\to\infty).$$

This proves the lemma.

REMARK. The conclusion of the lemma is incorrect for arbitrary slowly varying functions.

Recently Erdős and Segal constructed a function (see [1]) such that $\Sigma_{m\leqslant x}(\mu(m)/m)L(x/m)\neq o(L(x))\ (x\to\infty)$.

LEMMA 2. *Suppose* $\Sigma_{m\leqslant x}n(x/m)/m=\int_1^x L(u)/u\,du+L(x)$ *where* L *satisfies the conditions of Lemma* 1. *Then* $n(x)\sim L(x)\ (x\to\infty)$.

PROOF. Möbius inversion and Lemma 1 give

$$n(x)=\sum_{m\leqslant x}\frac{\mu(m)}{m}\int_1^{x/m}\frac{L(u)}{u}\,du+\sum_{m\leqslant x}\frac{\mu(m)}{m}L\left(\frac{x}{m}\right)$$
$$=\int_1^x\frac{L(u)}{u}N\left(\frac{x}{u}\right)du+o(L(x))$$
$$=\int_1^x L\left(\frac{x}{v}\right)\frac{N(v)}{v}\,dv+o(L(x))\sim L(x)\int_1^\infty\frac{N(v)}{v}\,dv\sim L(x)$$
$$(x\to\infty)$$

by dominated convergence, since

$$\int_1^x\frac{1}{v}\sum_{k\leqslant v}\frac{\mu(k)}{k}\,dv=\sum_{k\leqslant x}\frac{\mu(k)}{k}\int_k^x\frac{dv}{v}$$
$$=\log x\sum_{k\leqslant x}\frac{\mu(k)}{k}-\sum_{k\leqslant x}\frac{\mu(k)}{k}\log k\to 1\qquad(x\to\infty)$$

(see [10]).

REMARK. The proof of Lemma 2 implies the following. If $h(x)=\int_1^x L(u)/u\,du+L(x)$ where L satisfies the conditions of Lemma 1, then $\Sigma_{m\leqslant x}(\mu(m)/m)h(x/m)\sim(\int_1^x s\,dh(s))/x$, which is the auxiliary function of $h(x)$. For functions $h(x)$ of the form $\int_1^x L(u)/u\,du$, L slowly varying the

conditions of Lemma 1 are not necessary for this result. Compare with Theorem 1 of [13].

The following lemma gives the same statement under different conditions on $L(x)$.

LEMMA 3. *Suppose* $\sum_{m\leqslant x} n(x/m)/m = \int_1^x L(u)/u\,du + L(x)$ *where* $L(x)\to\infty$ *and* $L(x)\in\Pi$. *Then* $n(x)\sim L(x)$ $(x\to\infty)$.

PROOF. We write $L(x) = \int_1^x \gamma(u)/u\,du + o(\gamma(x))$ where γ is slowly varying. Then Möbius inversion gives

$$n(x) = \sum_{m\leqslant x}\frac{\mu(m)}{m}\int_1^{x/m}\frac{L(u)+\gamma(u)}{u}\,du + \sum_{m\leqslant x}\frac{\mu(m)}{m}o\left(\gamma\left(\frac{x}{m}\right)\right).$$

The first part is asymptotic to $L(x)$ as in Lemma 2 since $\gamma(x) = o(L(x))$ $(x\to\infty)$. For the second term we proceed as follows. Suppose $|o(\gamma(x))| < \varepsilon\gamma(x)$ for $x \geqslant x_0$. Then we have

$$\left|\sum_{m\leqslant x}\frac{\mu(m)}{m}o\left(\gamma\left(\frac{x}{m}\right)\right)\right| \leqslant \varepsilon\sum_{m\leqslant x/x_0}\frac{1}{m}\gamma\left(\frac{x}{m}\right) + \sum_{x/x_0<m\leqslant x}\frac{1}{m}\left|o\left(\gamma\left(\frac{x}{m}\right)\right)\right|$$

$$\leqslant \varepsilon\sum_{m\leqslant x/x_0}\frac{1}{m}\gamma\left(\frac{x}{m}\right) + c\sum_{x/x_0<m\leqslant x}\frac{1}{m} < \varepsilon\sum_{m\leqslant x}\gamma\left(\frac{x}{m}\right)/m + O(1)$$

where $c = \sup_{x\in(1,x_0)}|o(\gamma(x))|$. Now we have $\sum_{m\leqslant x}\gamma(x/m)/m\sim L(x)$ $(x\to\infty)$ by Theorem 2. This proves the lemma.

THEOREM 3. *Suppose*

(i) $\int_1^x L(u)/u\,du + o(L(x)) = \sum_{m\leqslant x} n(x/m)/m$, *where* $L(x)\to\infty$ $(x\to\infty)$ *is slowly varying*,

(ii) $R(x)\equiv\sum_{m\leqslant x} n(x/m)/m$ *is nondecreasing*,

(iii) $L_*(x)\equiv(\int_1^x t\,dR(t))/x$ *is nondecreasing*,

(iv) $L_*(x)/L_*(x-1)\leqslant 1 + x^{-\alpha}$ *with* $\alpha>0$ *for* $x>x_0(\alpha)$, *then* $n(x)\sim L(x)$ $(x\to\infty)$.

PROOF. We have $R(x) = \int_1^x L_*(t)/t\,dt + L_*(x)$ where $L_*(x)$ is defined as above and $L_*(x)\sim L(x)$ (see [3]) satisfies the conditions of Lemma 2. This proves the theorem.

COROLLARY 2. *Combination with Theorem* 2 *yields that with the assumptions of Theorem* 3 *and* $n(x)/x$ *integrable we have*

$$\int_1^x\frac{n(t)}{t}\,dt = \int_1^x\frac{L(u)}{u}\,du - \gamma L(x) + o(L(x)) \qquad (x\to\infty).$$

This is the same conclusion as in the papers of Segal and Jukes.

THEOREM 4. *Suppose* (i) *and* (ii) *of Theorem* 3 *and* $L_*(x)\in\Pi$. *Then* $L(x)\sim n(x)$ $(x\to\infty)$.

PROOF. We use now Lemma 3. The rest of the proof is as in Theorem 3.

Combining Corollary 1 with Theorems 3 or 4 gives the following Tauberian theorem.

THEOREM 5. *If* $\tilde{n}(1/s) = \int_1^s L(u)/u\,du + o(L(s))$ $(s \to \infty)$ *with* $L(s) \to \infty$ *slowly varying and* n: $R^+ \to R^+$ *satisfies the conditions*

(i) $n(u)/u$ *is integrable on* $(0, R)$ *for every* $R > 0$,

(ii) $R(x) \equiv \sum_{m \leqslant x} n(x/m)/m$ *is nondecreasing,*

(iii) $L_*(x) \equiv (\int_1^x t\,dR(t))/x$ *is nondecreasing,*

and

(iv) $L_*(x) \in \Pi$,

or

(v) $L_*(x)/L_*(x-1) \leqslant 1 + x^{-\alpha}$ *with* $\alpha > 0$ *for* $x \geqslant x_0(\alpha)$,

then

$$\frac{\displaystyle\int_1^s \frac{n(t)}{t}\,dt - \int_1^s \frac{L(t)}{t}\,dt}{L(s)} \to 0 \qquad (s \to \infty).$$

PROOF. Applying Corollary 1 and Theorems 3 or 4 we have $n(x) \sim L(x)$. Application of Theorem 1 yields the result.

For regularly varying functions the following analogue of Theorem 5 is well known.

THEOREM 6. *If* $\tilde{n}(1/s) \sim \zeta(\alpha+1)\Gamma(\alpha+1)s^\alpha L(s)$ *where* $L(s) \in RV_\alpha^{(\infty)}$ *with* $\alpha > 0$ *and* $n(u)$ *satisfies the following conditions*

(i) $n(u)/u$ *and* $(n(u)/u)\log u$ *are integrable on* $(0, R)$ *for every* $R > 0$,

(ii) $n(u)$ *is nondecreasing,*

then $n(u) \sim u^\alpha L(u)$ $(u \to \infty)$.

For a proof see [**9**, Theorem 2].

EXAMPLE 1. The function $g(x) = e^{\sqrt{\log x}}$ belongs to the class Π. This function does not satisfy the assumptions of Theorem B since the auxiliary function $xg'(x)$ is growing too fast. With the given conditions however we can apply Corollary 2 to get the desired result. On the other hand, for functions which belong to the class Π with a bounded auxiliary function our Tauberian results are not applicable.

2. The conditions on $L(x)$ in Lemmas 2 and 3 do not imply each other. This is shown by the following examples. If $L_1(x) = 2\log x + \sin\log x$, and $L_2(x) = [\log x] + \log^2(x+1)$, then L_1 (resp., L_2) satisfies the conditions of Lemma 2 (resp., 3) and not the conditions of Lemma 3 (resp., 2).

The author wishes to express his gratitude to Dr. L. de Haan for helpful and stimulating discussion which led to improvements on several parts of this paper.

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