ABSTRACT. Precise limits on the size of exceptional sets for which functions in the Lebesgue class $L^p$, can fail to be absolutely Abel summable are given in terms of Bessel capacity.

1. Introduction. In this note we present two theorems which generalize results of [1] and [2]. We do this in the context of the Lebesgue classes $L^p$ and the capacity theory developed in [3]. Here we introduce those facts from [3] which are for the most part contained in Theorem 16 of that paper.

Let $Q_k = \{x \in \mathbb{R}^k: -\frac{1}{2} < x_i < \frac{1}{2}, i = 1, 2, \ldots, k\}$ and $k > 2$ be the torus in $k$-dimensional Euclidean space. Let $0 < \alpha < k$, and $g_\alpha$ be the kernel of the Bessel potential which is given by the positive function whose Fourier coefficients are $\hat{g}_\alpha(n) = \left(1 + 4\pi^2|n|^2\right)^{-\alpha/2}$ where $n$ is a point in the $k$-dimensional integral lattice plane. Let $\mathcal{E}_1^+$ be the space of all nonnegative Radon measures of finite total variation on $Q_k$. If $\nu \in \mathcal{E}_1^+$, $\|\nu\|$ denotes its total variation. For $\nu \in \mathcal{E}_1^+$, $g_\alpha(\nu, x) = (g_\alpha * \nu)(x)$.

For $Z \subset Q_k$ an analytic set the capacity of $Z$ is defined for $1 < p < \infty$ by

$$c_{\alpha, p}(Z) = \sup\|\nu\|,$$

where the supremum is taken over all those $\nu \in \mathcal{E}_1^+$ concentrated on $Z$ for which $\|g_\alpha(\nu, \cdot)\|_{L^p} < 1$ and $p' = p/(p - 1)$.

If $Z$ has positive capacity then there is a nontrivial $\mu \in \mathcal{E}_1^+$ satisfying the variational problem of (1.1), the function $f$ defined by

$$f_p^{-1}(x) = (c_{\alpha, p}(Z))^{p-1} g_\alpha(\mu, x)$$

is in $L^p$, and $\|f\|_p = c_{\alpha, p}(Z)$. Moreover $\mu$ is concentrated on the set $Z \cap \{x: (f * g_\alpha)(x) = 1\} = Z_0$ and the set $Z - Z_0$ has zero $c_{\alpha, p}$ capacity. Such $f$ and $\mu$ are called capacitary distributions for $Z$. Let $f_0 = f/\|f\|_p$ and $\mu_0 = \mu/\|\mu\|$. Finally, for any function $h$ satisfying $(h * g_\alpha)(x) > 1$ on $Z$ and $\|h\|_p = 1$, and any measure $\nu$ concentrated on $Z$ satisfying $\|\nu\| = 1$, and $\|g_\alpha(\nu, \cdot)\|_{L^p} < 1$ we have

$$g_\alpha(\mu_0, h) < g_\alpha(\mu_0, f_0) < g_\alpha(\nu, f_0)$$

where $g_\alpha(\mu, f) = \int (f(\cdot)g_\alpha(x - \cdot)) \, d\mu(x) \, d\nu$.

We say a function $f$ belongs to the Lebesgue class $L^p_\alpha$ if $f$ can be written as $f_0 * g_\alpha$ for some $f_0$ in $L^p = L^p(Q_k)$.
2. Absolute Abel summability. In this section we deal with absolute Abel summability of multiple Fourier series. We say the Fourier series of a function $f$ is absolutely Abel summable at a point $x^0$ if

$$\int_0^1 \left| \frac{\partial f}{\partial t}(x^0, t) \right| dt < \infty$$

where

$$f(x, t) = \sum_{m} \hat{f}(m)e^{2\pi i m x - |m| t} \quad \text{for} \ t > 0.$$  

The theorems we prove are the following.

**Theorem 1.** Let $f$ be a function of class $C^p_\infty(Q_k)$. Then $f$ is absolutely Abel summable except possibly on a set of zero $c_{\alpha, p}$ capacity.

**Theorem 2.** Let $Z$ be a closed set in $Q_k$ which is of zero $c_{\alpha, p}$ capacity. Then there exists a function $f$ in $C^p_\infty(Q_k)$ such that

$$\int_0^1 \left| \frac{\partial f(x, t)}{\partial t} \right| dt = \infty \quad \text{for each} \ x \in Z.$$  

It should be noted that in [2] these theorems are given for the case $p = 2$ and ordinary capacity, which is motivated by one dimensional results of [1].

Let

$$g_\alpha(x, t) = \sum_{m} \frac{e^{2\pi i m x - 2\pi |m| t}}{(1 + 4\pi^2 |m|^2)^{\alpha/2}}, \quad t > 0.$$  

We have the following which can be obtained from an application of the Poisson summation formula and the properties of Bessel potentials listed in §7 of [3].

(2.1) For $\alpha > 0$, $g_\alpha(x, t) \to g_\alpha(x)$ provided $x$ is not a lattice point, as $t \to 0$.

(2.2) The function $g_\alpha(x)$ is continuous if $x$ is not a lattice point, $g_\alpha \in L^1$ and $g_\alpha > 0$.

$$\int_0^1 \left| \frac{\partial g_\alpha}{\partial t}(x, t) \right| dt \leq g_\alpha(x) + c(k) \quad \text{where} \ c(k) = \sum_{m} e^{-2\pi |m|}.$$  

To see that (2.3) holds, note that

$$\int_0^1 \left| \frac{\partial g_\alpha}{\partial t}(x, t) \right| dt \leq \lim_{t \to 0} \int_0^1 \left| \frac{\partial g_\alpha}{\partial t}(x, t) \right| dt,$$

and begin by applying the Poisson summation formula to justify that

$$\frac{\partial g_\alpha}{\partial t}(x, t) = \int_{y \in \mathbb{R}^k} g_\alpha(x - y) \frac{\partial P_t}{\partial t}(y) \, dy$$

where $P_t(y) = C \frac{t}{(t^2 + |y|^2)^{(k+1)/2}}$.

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The positivity of $g_a$ is used several times in the following. It follows that

$$\int_1^1 \left| \frac{\partial g_a(x, t)}{\partial t} \right| dt < \int_e^1 \int g_a(x - y) \left| \frac{\partial P_t}{\partial t} (y) \right| dy \ dt$$

$$\leq \left\{ \int_e^1 \int_{|y| < 2\sqrt{k}} + \int_e^1 \int_{|y| > 2\sqrt{k}} \right\} g_a(x - y) \left| \frac{\partial P_t}{\partial t} \right| dy \ dt$$

$$\equiv A + B.$$ 

Direct calculation of $\partial P_t / \partial t$ shows that $\partial P_t(y) / \partial t > 0$ if and only if $|y| > \sqrt{k} t$. Define

$$l = l(y, \epsilon) = \begin{cases} 
\epsilon, & |y| < \epsilon \sqrt{k}, \\
1, & |y| > \sqrt{k}, \\
|y| / \sqrt{k}, & \epsilon \sqrt{k} < |y| < \sqrt{k}.
\end{cases}$$

Then it follows that

$$A = -\int_{|y| < 2\sqrt{k}} g_a(x - y) \int_1^1 \frac{\partial P_t}{\partial t} (y) \ dt \ dy$$

$$+ \int_{|y| < 2\sqrt{k}} g_a(x - y) \int_1^1 \frac{\partial P_t}{\partial t} (y) \ dt \ dy$$

$$= \int_{|y| < 2\sqrt{k}} g_a(x - y) \left[ P_t(y) - P_t(y) \right] \ dy$$

$$+ \int_{|y| < 2\sqrt{k}} g_a(x - y) \left[ P_1(y) - P_1(y) \right] \ dy$$

$$< \int_{|y| < 2\sqrt{k}} g_a(x - y) \left[ P_t(y) + P_1(y) \right] \ dy$$

$$< g_a(x, \epsilon) + \int_{|y| < 2\sqrt{k}} g_a(x - y) P_1(y) \ dy.$$ 

Similar considerations for $\partial P_t(y) / \partial t$ lead to

$$B = \int_{|y| > 2\sqrt{k}} g_a(x - y) \left[ P_t(y) - P_\epsilon(y) \right] \ dy < \int_{|y| > 2\sqrt{k}} g_a(x - y) P_1(y) \ dy.$$ 

Combining the estimates for $A$ and $B$ gives

$$\int_0^1 \left| \frac{\partial g_a(x, t)}{\partial t} \right| dt < \lim_{\epsilon \to 0} g_a(x, \epsilon) + g_a(x, 1).$$

Again by the Poisson summation formula

$$g_a(x, 1) < \left| \sum_{m} \frac{1}{(1 + 4\pi^2 |m|^2)^{\alpha/2}} e^{-2\pi |m| e^{im^x}} \right| < c(k).$$
Next, let
\[ f(x, t) = \sum_m \hat{f}(m)e^{2\pi im \cdot x - 2\pi |m| t} \]
\[ = \sum_m \hat{f}_0(m) \frac{1}{(1 + 4\pi^2|m|^2)^{\sigma/2}} e^{2\pi im \cdot x - 2\pi |m| t} \]
\[ = \int_{Q_k} f_0(y) \cdot g_\sigma(x - y, t) \, dy. \]

It follows that
\[ \frac{\partial f}{\partial t}(x, t) = \int_{Q_k} f_0(y) \frac{\partial g_\sigma}{\partial t}(x - y, t) \, dy \quad (2.4) \]

so that
\[ \left| \int_0^1 \frac{\partial f}{\partial t}(x, t) \, dt \right| \leq \int_{Q_k} |f_0(y)| \left| \int_0^1 \frac{\partial g_\sigma}{\partial t}(x - y, t) \, dt \right| \, dy \]
\[ \leq c(k) \int_{Q_k} |f_0(y)| \, dy + (|f_0| * g_\sigma)(x). \quad (2.5) \]

We now present the proof of Theorem 1.

**Proof of Theorem 1.** Let \( E_\infty = \{ x : |\int_0^1 \partial f(x, t) / \partial t \, dt | = \infty \} \) and suppose that \( E_\infty \) has positive \( c_{a, p} \) capacity. Then there exists a measure \( \mu \) (capacitary distribution) concentrated on \( B_\infty = \{ x : g_\sigma(x, f_\mu) = 1 \} \cap E_\infty \) where \( f_\mu(y) = c_{a, p}(E_\infty)(|g_\sigma(\mu, y)|/c_\sigma \mu) \). We use the notation compatible with §1:

\[ g_\sigma(\mu, y) = \int_{Q_k} g_\sigma(x - y) \, d\mu(x), \]
\[ g_\sigma(x, f_\mu) = \int_{Q_k} g_\sigma(x - y) f_\mu(y) \, dy, \]
\[ c_{a, p}(E_\infty) = \int f_\mu^p(y) \, dy = c_{a, p}(E_\infty) \int g_\sigma(\mu, y)^p \, dy, \]

and \( \| \mu \| = c_{a, p}(E_\infty) > 0 \).

In this case, let \( \nu = c_{a, p}^{-1}(E_\infty) \mu \) so that \( \| \nu \| = 1 \) and \( \nu(Q_k - E_\infty) = 0 \). Then

\[ \int \int |f_0(y)| g_\sigma(x - y, \nu) \, dy \, dx = \int |f_0(y)| \cdot g_\sigma(\nu, y) \, dy \]
\[ \leq \| f_0 \|_{p'} \cdot \| g_\sigma(\nu, \cdot) \|_{p'} = \| f_0 \|_{p'} \cdot c_{a, p}^{-1}(E_\infty). \quad (2.6) \]

The last inequality follows from the fact that \( \| g_\sigma(\mu, \cdot) \|_{p'} < 1 \).

We now have that
\[ \int_{Q_k} \int_0^1 \left| \frac{\partial f}{\partial t}(x, t) \right| \, dt \, dx = \infty, \quad (2.7) \]
while on the other hand, by (2.5) and (2.6), we have
\[
\int_{Q_k} \int_0^1 \frac{\partial f}{\partial t}(x, t) \ dt \ dv(x) \leq c(k)|Q_k|^{1/p'}\|f_0\|_p + \int |f_0(y)|g_a(x - y) \ dv(x) \ dy
\]
\[
= \left[ c(k)|Q_k|^{1/p'} + c_{a,p}^{-1}(E_\infty) \right] \|f_0\|_p < \infty. \quad (2.8)
\]
This contradiction completes the proof of Theorem 1.

3. Counterexample. We now give the construction of the counterexample for Theorem 2.

Let \( Z \) be a closed set in \( Q_k \) such that \( c_{a,p}(Z) = 0 \). Since \( c_{a,p} \) is a capacity, we can approximate \( c_{a,p}(Z) \) by \( c_{a,p}(K_p) \) where \( K_p = \{ x : \text{dist}(x, Z) < \rho \} \); that is,
\[
c_{a,p}(Z) = \lim_{\rho \to 0} c_{a,p}(K_p). \quad (3.1)
\]
For each \( \rho > 0 \) let \( \mu_\rho \) and \( f_\rho \) be the capacitary distributions for \( K_p \) with \( f_\rho(y)^{p-1} = c_{a,p}(K_p)^{p-1}g_a(\mu_\rho, y) \) almost everywhere and \( f_\rho \ast g_a = 1 \) almost everywhere with respect to capacity \( c_{a,p} \) on \( K_p \). Then it follows that
\[
\int g_a(x - y) \left( \int g_a(y - z) \ d\mu_\rho(z) \right)^{1/(p-1)} \ dy = c_{a,p}(K_p)^{-1} \quad (3.2)
\]
almost everywhere in \( K_p \) with respect to \( c_{a,p} \) capacity. We choose a sequence \( \{ \mu_\rho \} \) associated in the above fashion such that \( c_{a,p}(K_p) = \| \mu_\rho \| \to 0 \).

Then \( \| f_\rho \|_p = c_{a,p}(K_p) \). Note that \( f_\rho \) is not constant on \( K_p \) but it has a potential which is equal to one almost everywhere with respect to \( c_{a,p} \) capacity. Let
\[
f_0^k = \sum_{j=1}^k c_j f_\rho \quad \text{and} \quad f^k = \sum_{j=1}^k g_a \ast (c_j f_\rho). \quad (3.3)
\]
Hence \( f^k \) is in \( C_0^k(Q_k) \). For \( c_j \) use \( j^{-\delta}c_{a,p}(K_\rho)^{-1} \) with \( \delta > 1 \) so that we have \( \| f_0^k \|_p < \sum_{j=1}^k j^{-\delta} < \infty \) and \( \{ f_0^k \} \) forms a Cauchy sequence in \( L^p(Q_k) \). Let \( f_0 \) denote the limit of \( f_0^k \) as \( k \) tends to infinity. Since \( f_\rho \) is a positive function, \( f_0 \) is also the pointwise limit of \( f_0^k \). Furthermore, for almost every \( x \) in \( K_\rho \) with respect to \( c_{a,p} \) capacity, we have
\[
(f_0^k \ast g_a)(x) > \sum_{j=1}^k j^{-\delta}c_{a,p}(K_\rho)^{-1}(f_\rho \ast g_a)(x) = \sum_{j=1}^k j^{-\delta}c_{a,p}(K_\rho)^{-1}. \quad (3.4)
\]
We choose the sequence \( \{ \rho_j \} \) such that \( c_{a,p}(K_\rho) < j^{-\delta} \). Then in \( K_\rho \), \( (P_t \ast f_0^k \ast g_a) \) tends to \( \sum_{j=1}^k j^{-\delta}c_{a,p}(K_\rho)^{-1} > k \) as \( t \) tends to zero, since \( f_0^k \ast g_a \) is essentially constant in \( K_\rho \).

Since
\[
|f(x, 1) - f(x, t)| < \int_0^1 \left| \frac{\partial f}{\partial s}(x, s) \right| \ ds \quad \text{and} \quad |f(x, 1)| < c(k)|Q_k|^{1/p'},
\]
it will follow that
\[ \int_0^1 \left| \frac{\partial f}{\partial t}(x, t) \right| \, dt = \infty \quad \text{for } x \in Z \] (3.5)
if we show
\[ \lim_{t \to 0} f(x, t) = \infty \quad \text{for } x \in Z. \] (3.6)
But for \( x \) in \( Z \) we have
\[ \lim_{t \to \infty} \inf f(x, t) \geq \lim_{t \to \infty} \inf (P_t \ast f \ast g) > k \]
for all \( k \). This concludes the construction of Theorem 2.

REFERENCES


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