ON TWO RESULTS OF J. DUGUNDJI ABOUT EXTENSIONS
OF MAPS AND RETRACTIONS

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Abstract. We give a short proof of Dugundji's result that spheres in
infinite-dimensional normed spaces are absolute retracts.

In [1], Dugundji proved a very important extension theorem and used it to
show the following fact:

THEOREM 1. Let L be a normed linear space, B := \{x \in L| ||x|| < 1\} and
C := \{x \in L| ||x|| = 1\}. If C is not compact, then C is a retract of B (and
hence an absolute retract).

An important consequence of Theorem 1 for the applications is that C is
contractible and an absolute extensor for metrizable spaces (cf. [3]). Another
well-known consequence is the fact that closed balls in infinite-dimensional
normed spaces do not have the fixed point property for continuous maps.

The aim of this paper is to give a slightly sharper version of the Dugundji
extension theorem, which allows a very short and intuitive proof of Theorem
1.

THEOREM 2. Let X be a metric space, A \subset X a closed subset, D a dense
subset of A, and L a normed space (or more generally a locally convex space or
an affine space of type m, cf. [2]). Then each continuous f: A \rightarrow L has a
continuous extension F: X \rightarrow L with F(X) \subset f(A) \cup \{convex hull of f(D)\}.

The proof of Theorem 2 is the same as the one of the original theorem, if
one chooses the points a_i \in D (cf. [2, p. 188]).

Proof of Theorem 1. Since C is not compact, L is infinite-dimensional.
Hence L has a proper dense linear subspace L'.\footnote{One should recall that it is easy to prove the existence of such a space L': Any unbounded real-valued function on a normalized Hamel basis of L uniquely determines an unbounded linear functional f on L. Then L' := \{kernel of f\} is not closed and codim L' = 1, and hence L' is dense.} By Theorem 2, applied to
X := B, A := C, D := C \cap L' and f := id_C, there exists a continuous map
F: B \rightarrow L with F|_C = id_C and F(B) \subset C \cup \{convex hull of C \cap L'\} = C \cup
(B \cap L') \subsetneq B. Choose x_0 \in B \setminus F(B), and let r: B \setminus \{x_0\} \rightarrow C be the radial
retraction. Then r \circ F is a retraction from B to C.

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