SIGN COMPATIBLE EXPRESSIONS FOR MINORS OF THE MATRIX $I - A$

BY

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Abstract. Let $A = (a_{ij})$ be an $n \times n$ nonnegative matrix having row sums less than or equal to one. This paper shows that the $ij$th minor of $I - A$ can be expressed as

$$(-1)^{i+j} \sum \prod_{k \neq s} r_k a_{pq}$$

where

$$r_k = 1 - \sum_{s=1}^{n} a_{ks}$$

and each $\prod_{k \neq s} a_{pq}$ is a product of exactly $n - 1$ numbers taken from $r_k, a_{pq}$ for $k, p, q = 1, \ldots, n$. This theorem is then used to obtain perturbation results concerning the matrix $I - A$.

Perturbation results in matrix theory are concerned with estimating the error in matrix computations. This paper provides perturbation results for the matrix $I - A$ where $A = (a_{ij})$ is nonnegative having row sums less than or equal to one. The method by which these perturbation results are achieved is a variant of that given by Sengupta [2] in his work on comparing stochastic eigenvectors of two irreducible stochastic matrices. The method, as we apply it, first gives expressions for the minors of $I - A$, in terms of the entries of $A$, and then uses these expressions to produce useful perturbation results for this matrix.

The theorem of the paper produces expressions for the minors of $I - A$.

Theorem. Let $A$ be an $n \times n$ nonnegative matrix having largest row sum less than or equal to one. Then

$$|(I - A)_{ij}| = (-1)^{i+j} \sum \prod_{k \neq s} r_k a_{pq}$$

where

$$r_k = 1 - \sum_{s=1}^{n} a_{ks}$$

and each $\prod_{k \neq s} a_{pq}$ is a product of exactly $n - 1$ numbers taken from $r_k, a_{pq}$ for $k, p, q = 1, \ldots, n$. 

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PROOF. The method of proof is induction on $n$. The case $n = 2$ can be proved by checking all choices for $i$ and $j$. Thus, suppose the theorem is true for all $n \times n$ matrices $A$, satisfying the hypothesis, where $n < n_1$. Now let $A$ be an $n \times n$ matrix, satisfying the hypothesis, where $n = n_1$. The argument is divided into two cases. The first case to be considered is when $i \neq j$. Here we will assume $i < j$ as the case $i > j$ is argued similarly.

For this case, we first define an $n \times n$ matrix $E_{pq} = (e_{rs})$ where

$$e_{rs} = \begin{cases} 1, & \text{if } r = s \text{ and } r \neq p, r \neq q, \\ 1, & \text{if } r = p \text{ and } s = q, \\ 1, & \text{if } r = q \text{ and } s = p, \\ 0, & \text{otherwise}. \end{cases}$$

Let $P = E_{12}E_{23} \cdots E_{i-1,i}$, a permutation matrix, and set $P(I - A)P' = I - PAP' = I - B$. Then

$$|(I - A)_{ij}| = (-1)^{i-1} |(I - B)_{ij}|.$$ 

Hence we need only prove the result for $i = 1$ and $j > 1$.

For this then we expand $|(I - A)_{ij}|$ about the 1st column achieving that

$$|(I - A)_{ij}| = \sum_{s > 1} (-1)^{s-1}(-a_{s1}) |[(I - A)_{ij}]*|,$$

where $[(I - A)_{ij}]_{s1}$ denotes the matrix obtained from $(I - A)$ by deleting rows $1, s$ and columns $j, 1$.

Now noting that $[(I - A)_{ij}]_{s1} = [(I - A)_{11}]_{s1}$ and applying the induction hypothesis yields that

$$|(I - A)_{ij}| = \sum_{s > 1} (-1)^{s+1}a_{s1} |[(I - A)_{11}]_{s1}|$$

$$= \sum_{s > 1} (-1)^{s+1}a_{s1}((-1)^{s+1}\sum (r_k + a_{k1})a_{pq})$$

$$= (-1)^{i+1}\sum \prod r_k a_{pq}$$

where each $\prod r_k a_{pq}$ is a product of exactly $n - 1$ numbers taken from $r_k, a_{pq}$ for $k, p, q = 1, \ldots, n$.

For the case $i = j$, we assume without loss of generality that $i = j = 1$. Write

$$(I - A)_{11} = \begin{bmatrix} r_2 + \sum_{s \neq 2} a_{2s} & -a_{23} & \cdots & -a_{2n} \\ -a_{32} & r_3 + \sum_{s \neq 3} a_{3s} & \cdots & -a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n2} & -a_{n3} & \cdots & r_n + \sum_{s \neq n} a_{ns} \end{bmatrix}.$$
Adding columns two through \( n - 1 \) to column one yields

\[
B = \begin{pmatrix}
  r_2 + a_{21} & -a_{23} & \cdots & -a_{2n} \\
  r_3 + a_{31} & r_3 + \sum_{j \neq 3} a_{3j} & \cdots & -a_{3n} \\
  \cdots & \cdots & \cdots & \cdots \\
  r_n + a_{n1} & -a_{n3} & \cdots & r_n + \sum_{j \neq n} a_{nj}
\end{pmatrix}
\]

Expanding the determinant about the first column of \( B \) and noting that \( B_{s1} = [(I - A)_{s1}]_{s+1,2} \) yields, by the induction hypothesis, that

\[
\det(I - A)_{s1} = \sum_{s=1}^{n-1} (-1)^{s+1} (r_{s+1} + a_{s+1,1}) \det(I - A)_{s+1,2}
= \sum_{s=1}^{n-1} (-1)^{s+1} (r_{s+1} + a_{s+1,1}) \left( -1 \right)^{s+1} \left( \sum (r_k + a_{k1})a_{pq} \right)
= \sum \prod r_k a_{pq}
\]

where each \( \prod r_k a_{pq} \) is a product of exactly \( n - 1 \) numbers taken from \( r_k, a_{pq} \) for \( k, p, q = 1, \ldots, n \).

This theorem is now applied to yield our first perturbation result. This result estimates the error in computing \( (I - A)^{-1} \), when it exists.

**Corollary 1.** Let \( A \) and \( \hat{A} \) be \( n \times n \) nonnegative matrices with row sums less than or equal to one and having spectral radius less than 1. Set \( B = (I - A)^{-1} \) and \( \hat{B} = (I - \hat{A})^{-1} \).

(1) \( a_{ij} < \theta \hat{a}_{ij}, r_k < \theta \hat{r}_k \) and
(2) \( \hat{a}_{ij} < \theta a_{ij}, \hat{r}_k < \theta r_k \)
then, for all \( b_j \neq 0 \), \( (\hat{b}_j - b_j) / b_j < \theta^{n-1} \theta^n - 1 \).

**Proof.** Note that

\[
b_j = \frac{(-1)^{i+j} \det(I - A)_{ij}}{|I - A|} \quad \text{and} \quad \hat{b}_j = \frac{(-1)^{i+j} \det(I - \hat{A})_{ij}}{|I - \hat{A}|}.
\]

Thus, if \( b_j \neq 0 \), application of the theorem yields that

\[
\frac{\hat{b}_j}{b_j} = \frac{|(I - \hat{A})_{ij}|}{|I - A|} < \frac{\theta^{n-1}|(I - A)_{ij}|}{|I - A|} < \theta^{n-1} \theta^n.
\]

Hence,

\[
\frac{\hat{b}_j - b_j}{b_j} < \theta^{n-1} \theta^n - 1.
\]

A second perturbation result estimates the error in solving Leontif's open economic model.
COROLLARY 2. Let $A$ and $\hat{A}$ be $n \times n$ nonnegative matrices having largest row sums less than or equal to one and having spectral radii less than one. Let $b$ and $\hat{b}$ be $1 \times n$ nonnegative vectors with $x(I - A) = b$ and $\hat{x}(I - \hat{A}) = \hat{b}$. If

1. $a_\ell < \hat{a}_\ell$, $\rho_k(A) < \hat{\rho}_k(\hat{A})$, $b_\ell < \hat{b}_\ell$ and
2. $\hat{a}_\ell < \theta a_\ell$, $\rho_k(\hat{A}) < \theta \rho_k(A)$, $\hat{b}_\ell < \theta b_\ell$

then, for all $x_i \neq 0$, $(\hat{x}_i - x_i)/x_i < (\theta \hat{\theta})^n - 1$.

PROOF. Note first that $x = (I - A)^{-1}b$ and $\hat{x} = (I - \hat{A})^{-1}\hat{b}$. Then, if $x_i \neq 0$, application of the theorem yields that

$$\hat{x}_i = \frac{\sum \alpha_{i\ell}(I - \hat{A})_\ell b_\ell}{\prod_{\ell=1}^{n-1} |(I - \hat{A})_\ell|} < (\theta \hat{\theta})^n.$$

Hence

$$\frac{\hat{x}_i - x_i}{x_i} < (\theta \hat{\theta})^n - 1. \quad \square$$

The last perturbation result estimates the error in computing stochastic eigenvectors for stochastic matrices.

COROLLARY 3. Let $A$ and $\hat{A}$ be $n \times n$ irreducible stochastic matrices. Suppose $a$ and $\hat{a}$ are stochastic eigenvectors, belonging to one, for $A$ and $\hat{A}$ respectively. If

1. $a_\ell < \hat{a}_\ell$ and
2. $\hat{a}_\ell < \theta a_\ell$

then $(\hat{a}_i - a_i)/a_i < (\theta \hat{\theta})^n - 1$.

PROOF. First note that the Perron-Frobenius theory [1] gives that if $a$ and $\hat{a}$ are stochastic eigenvectors, belonging to one, for $A$ and $\hat{A}$ respectively, then $a$ and $\hat{a}$ are the unique solutions to

$$a(I - A) = 0 \quad \text{with} \quad \sum a_i = 1 \quad \text{and} \quad \hat{a}(I - \hat{A}) = 0 \quad \text{with} \quad \sum \hat{a}_i = 1.$$

Further, by the Perron-Frobenius theory, rank $(I - A) = \text{rank}(I - \hat{A}) = n - 1$ with the first $n - 1$ columns of both $I - A$ and $I - \hat{A}$ being linearly independent. Hence, the above equations are equivalent to

$$a[(I - A)_n e] = e_n \quad \text{and} \quad \hat{a}[(I - \hat{A})_n e] = e_n$$

where $(I - A)_n$ and $(I - \hat{A})_n$ are obtained by deleting the $n$th column of both $I - A$ and $I - \hat{A}$ respectively. Further $e_i$ is the $(0, 1)$-vector having its only nonzero entry in the $i$th position and $e = e_1 + \cdots + e_n$. Now, by Cramer's rule

$$a_i = (-1)^{i+n} \det((I - A)_n e)_{in}/\det((I - A)_n e)$$

and

$$\hat{a}_i = (-1)^{i+n} \det((I - \hat{A})_n e)_{in}/\det((I - \hat{A})_n e).$$

Thus

$$\frac{\hat{a}_i}{a_i} = \frac{\det((I - \hat{A})_n e)_{in}\det((I - A)_n e)}{\det((I - A)_n e)_{in}\det((I - \hat{A})_n e)}.$$
Noting that $\det((I - \hat{A})_n e)_m = \det((I - \hat{A})_n)$ and $\det((I - A)_n e)_m = \det((I - A)_n)$ and expanding $\det((I - A)_n e]$ and $\det((I - \hat{A})_n e]$ about the last column yields, by applying the theorem, that

$$\frac{\hat{a}_i}{\alpha_i} < \left( \theta^{n-1} \sum \prod r_k a_{pq} \right) \left( \sum \prod \hat{r}_k \hat{a}_{pq} \right) < (\theta \hat{\theta})^{n-1}.$$  

Hence,

$$\frac{\hat{a}_i - \alpha_i}{\alpha_i} < (\theta \hat{\theta})^{n-1} - 1. \quad \square$$

REFERENCES


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