ON THE CLASSIFICATION OF FINITE SIMPLE GROUPS
BY THE NUMBER OF INVOLUTIONS

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Abstract. Simple groups with \( k \) involutions, where \( k \equiv 1 \) (modulo 4), are completely determined.

The aim of this note is to prove the following:

Theorem. Let \( G \) be a finite simple group with \( I \) involutions and suppose that \( I \equiv 1 \) (mod 4). Then one of the following holds:

(a) \( I = 1 \) and \( G \) is cyclic of order 2,
(b) \( I = 105 \) and \( G \cong A_7 \),
(c) \( I = 165 \) and \( G \cong M_{11} \),
(d) \( I = \frac{q(q+\epsilon)}{2} \), and \( G \cong \text{PSL}(2, q) \), where \( q = p^n > 3 \) is a power of an odd prime, \( \epsilon = 1 \) or \( -1 \) and \( q \equiv \epsilon \) (mod 8),
(e) \( I = q^2(q^2 + q + 1) \) and \( G \cong \text{PSL}(3, q) \), where \( q = p^n \) is a power of an odd prime and \( q \equiv -1 \) (mod 4),
(f) \( I = q^2(q^2 - q + 1) \) and \( G \cong \text{PSL}(3, q) \), where \( q = p^n \) is a power of an odd prime and \( q \equiv 1 \) (mod 4).

Proof. By [4], a Sylow 2-subgroup of \( G \) is cyclic, generalized quaternion, dihedral of order \( > 8 \) or quasi-dihedral. In the cyclic case we get (a). A generalized quaternion Sylow 2-subgroup is impossible by [2] and in the dihedral or quasi-dihedral cases we get (b)–(f) by [3] and [1].

It is easy to check the following:

Corollary. Each of the above mentioned simple groups is characterized by the number of its involutions. In particular, \( M_{11} \) is the unique simple group with 165 involutions and \( A_7 \) is the unique simple group with 105 involutions.

Added in Proof. The groups \( A_8 \) and \( \text{PSL}(3, 4) \) are of the same order and each has 315 involutions. Conjecture: if two simple groups have the same number of involutions, then they are of the same order.

Bibliography


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