A UNIQUENESS THEOREM FOR A BOUNDARY VALUE PROBLEM

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ABSTRACT. In this paper it is proved that the two-point boundary value problem, namely \((d(4)/dx^4 + f)y = g, y(0) - A_1 = y(1) - A_2 = y''(0) - B_1 = y''(1) - B_2 = 0\), has a unique solution provided \(\inf_{x} f(x) = -\pi^4\). The given boundary value problem is discretized by a finite difference scheme. This numerical approximation is proved to be a second order convergent process by establishing an error bound using the \(L_2\)-norm of a vector.

1. Introduction. Consider the real two-point linear boundary problem

\[ Ly = \left[ d^4/dx^4 + f(x) \right] y = g(x), \quad 0 < x < 1, \]
\[ y(0) = A_1, \quad y(1) = A_2, \quad y''(0) = B_1, \quad y''(1) = B_2, \quad (1) \]

where the functions \(f(x)\) and \(g(x) \in C[0, 1]\). A more general problem of the form

\[ Ly = g(x), \quad y(a) = A_1, \quad y(b) = A_2, \quad y''(a) = B_1, \quad y''(b) = B_2 \]

can always be transformed into (1) by means of a substitution of the form \(X = (x - a)/(b - a)\). Problems of the form (1) frequently occur in plate deflection theory (see Reiss et al. [6]). The analytical solution of (1) is given by Timoshenko and Woinowsky-Krieger [7] provided the functions \(f(x)\) and \(g(x)\) are constants. In the general case we resort to some numerical techniques. Usmani and Marsden [8] have analyzed a second order convergent finite difference method for (1). Following this, Jain et al. [4] have developed and analysed higher order methods. The problem (1) does not always have a unique solution for all choices of \(f(x)\) as is apparent from the example

\[ y^{(4)} - \pi^4 y = 0, \quad y(0) = y(1) = y''(0) = y''(1) = 0 \]

which has as its solution \(y(x) = C \sin (\pi x)\) for arbitrary values of \(C\). The purpose of this note is to establish a sufficient condition that guarantees a unique solution for (1).

2. A uniqueness theorem. We shall give an elementary proof of the following theorem.
Theorem 1. The boundary value problem (1) has a unique solution provided
\[ \inf_x f(x) = -\eta > -\pi^4, \text{ that is } -f(x) < \eta. \] (2)

We preface the proof of this theorem with the following lemmas.

Lemma 2. If \( y(x) \in C^1[0, 1] \) and \( y(0) = y(1) = 0 \), then
\[ \pi^2 \int_0^1 y^2(x) \, dx < \int_0^1 [y'(x)]^2 \, dx. \]

Let \( C[0, 1] \) consist of all continuous functions on the interval \( I = [0, 1] \) and define in this section only \( \|y\| = \sup_x |y(x)|, \, x \in I \).

Lemma 3. If \( y(0) = y(1) = 0 \) and \( y(x) \in C[0, 1] \), then
\[ \|y\| < 0.5 \left[ \int_0^1 \left( y'(x) \right)^2 \, dx \right]^{1/2}. \]

For the proofs of these lemmas the reader should consult Hardy et al. [2, Theorem 256, p. 182] and Lees [5].

Lemma 4. For the differential system
\[ Ly = g(x), \quad y(0) = y(1) = y''(0) = y''(1) = 0, \]
\[ \|y\| < 0.5\pi \|g\| / \left[ \pi^4 - \eta \right]. \]

Proof. The system
(i) \[ y''(x) = z(x), \quad y(0) = y(1) = 0, \]
(ii) \[ z''(x) + f(x)y = g(x), \quad z(0) = z(1) = 0 \] (3)
is equivalent to the differential system of the theorem. On multiplying (3.i) by \( y(x) \) and integrating the result from 0 to 1, we find
\[ - \int_0^1 (y')^2 \, dx = \int_0^1 yz \, dx. \]

Now using the Cauchy-Schwartz inequality we obtain from the preceding equation
\[ \int_0^1 (y')^2 \, dx < \left[ \int_0^1 y^2 \, dx \right]^{1/2} \left[ \int_0^1 z^2 \, dx \right]^{1/2}. \]

On using Lemma 2, we derive from the preceding inequality
\[ \left[ \int_0^1 (y')^2 \, dx \right]^{1/2} < \frac{1}{\pi^2} \left[ \int_0^1 (z')^2 \, dx \right]^{1/2}. \] (4)

In a similar manner, from (3.ii), we derive
\[ \left[ \int_0^1 (z')^2 \, dx \right]^{1/2} < \left[ \pi^3 \frac{\|g\|}{\pi^4 - \eta} \right]. \] (5)
provided \( \eta \) satisfies (2). Now from (4) and (5) it follows that

\[ \left[ \int_0^1 (y')^2 \, dx \right]^{1/2} < \pi \frac{||g||}{\pi^4 - \eta}. \]

Lemma 4 now follows from (6) and Lemma 3.

**Proof of Theorem 1.** Assume that there exist two distinct functions \( u(x) \) and \( v(x) \) satisfying (1). Then it is easily seen that \( \phi(x) = u(x) - v(x) \) satisfies

\[ L\phi = 0, \quad \phi(0) = \phi(1) = \phi''(0) = \phi''(1) = 0. \]

Now, from Lemma 4 and (7) it follows that \( ||\phi|| < 0 \), which proves \( ||\phi|| \equiv 0 \) and \( u(x) \equiv v(x), x \in I \). This proves that the boundary value problem (1) has at most one solution.

In order to prove that (1) indeed has a solution, we define functions \( y_i(x) \), \( i = 1, \ldots, 4 \), as solutions of the respective initial value problems.

1. \( Ly_1 = g(x), \quad y_1(0) = A_1, \quad y_1'(0) = y_1''(0) = y_1'''(0) = 0, \)
2. \( Ly_2 = 0, \quad y_2'(0) = 1, \quad y_2(0) = y_2'(0) = y_2''(0) = 0, \)
3. \( Ly_3 = 0, \quad y_3'(0) = B_1, \quad y_3(0) = y_3'(0) = y_3''(0) = 0, \)
4. \( Ly_4 = 0, \quad y_4''(0) = 1, \quad y_4(0) = y_4'(0) = y_4''(0) = 0. \)

From the continuity of \( f(x) \) and \( g(x) \) we are assured that unique solutions of these initial value problems exist on \([0, 1]\). Furthermore the function \( z(x) = z(x, s, t) = y_1 + sy_2 + y_3 + ty_4, s, t \) being scalars, satisfies the initial value problem

\[ Lz = g(x), \quad z(0) = A_1, \quad z'(0) = s, \quad z''(0) = B_1, \quad z'''(0) = t. \]

The function \( z(x) \) will be a solution of (1) provided \( s, t \) satisfy

\[ sy_2(1) + ty_4(1) = A_2 - y_1(1) - y_3(1), \]
\[ sy_2'(1) + ty_4'(1) = B_2 - y_1'(1) - y_3'(1). \]

If \( \Delta = y_2(1)y_4'(1) - y_2'(1)y_4(1) \neq 0 \), a unique solution of the preceding linear system can be found, say \( s^*, t^* \), and the corresponding function \( z(s, s^*, t^*) \) then is the unique solution of (1). However, if \( \Delta = 0 \), then

\[ y_2(1)/y_2'(1) = y_4(1)/y_4'(1) = p \quad (\text{constant}). \]

We can assume that \( p \neq 0 \), because if \( p = 0 \), then \( y_2(1) = 0 \) and the solution of

\[ Ly_2 = 0, \quad y_2'(0) = y_2''(0) = y_2'''(0) = y_2(1) = 0 \]

from Taylor series has the property that \( y_2'(0) = 0 \), contradicting the original assumption that \( y_2'(0) = 1 \). Similarly \( p \) cannot be unbounded. Thus it follows that \( y_2(1) = py_2'(1), p < \infty \).

Now using the system (8.ii), and the Taylor series, we obtain

\[ y_2(1) = 1 - \frac{1}{2\pi}f(\alpha)y_2(\alpha), \quad 0 < \alpha < 1, \]
\[ y_2'(1) = -0.5f(\beta)y_2'(\beta), \quad 0 < \beta < 1. \]
On combining \( y_2(1) = py_2'(1) \) with equations (9) we obtain

\[
f(\alpha)y_2(\alpha) - 12pf(\beta)y_2(\beta) = 24,
\]
for all \( f(x) \in C \). In an attempt to determine \( y_2(\alpha) \) and \( y_2(\beta) \), we choose \( f(x) \equiv 1 \) and \( f(x) \equiv -1 \), giving the system

\[
y_2(\alpha) - 12py_2(\beta) = 24, \quad -y_2(\alpha) + 12py_2(\beta) = 24.
\]
But this latter system in the unknowns \( y_2(\alpha) \) and \( y_2(\beta) \) is inconsistent. We thus conclude that \( \Delta \) cannot vanish and the proof of the Theorem 1 is completed.

3. A discrete boundary value problem. Let \( N \) be a positive integer and \( h = (N + 1)^{-1} \). We define the grid points \( x_n = a + nh, \ n \in \{0, N + 1\} \cup S \) where \( S = \{1, 2, \ldots, N\} \). We denote by \( \Phi \) the set of all real-valued functions defined on \( \{x_n\}, \ n \in S \). Clearly \( \Phi \) is a real linear space of dimension \( N \). Also let \( ||u|| = \left[ \sum u_n^2 \right]^{1/2} \), where \( u_n \equiv u(x_n) \). Note that \( || \cdot || \) defines the \( L_2 \)-norm of a vector, a natural definition of a norm on vectors since this norm converges to \( \int_0^1 u^2(x) \, dx \) as \( h \to 0 \). We also have \( ||u|| = \sqrt{\mu} ||u||_2 \) where \( || \cdot ||_2 \) is the Euclidean norm (see Isaacson and Keller [3]). For a given matrix \( A = (a_{ij}) \), the matrix norm induced by the Euclidean vector norm we define the Hilbert or spectral norm of a matrix by \( ||A||_2 = \sqrt{\mu} \) where \( \mu \) is the largest eigenvalue of \( A^*A \) Here the operation * denotes the conjugate transpose of a matrix.

We now discretize the problem (1) by the following finite difference scheme

(i) \[-2y(x_0) + 5y(x_1) - 4y(x_2) + y(x_3) = -h^2y''(x_0) + h^4\left[ -\frac{1}{12}y^{(4)}(x_0) + y^{(4)}(x_1) \right] + t_1,\]

(ii) \[\delta^4y(x_n) = h^4y^{(4)}(x_n) + \frac{1}{6}h^6y^{(6)}(\omega_n), \quad n = 2, \ldots, N - 1, \quad x_{n-2} < \omega_n < x_{n+2},\]

(iii) \[y(x_{N-2}) - 4y(x_{N-1}) + 6y(x_N) - 2y(x_{N+1}) = -h^2y''(x_{N-1}) + h^4\left[ y^{(4)}(x_N) - \frac{1}{12}h^4y^{(6)}(x_{N+1}) \right] + t_N, \quad (10)\]

where \( t_i = \frac{59}{360} h^6y^{(6)}(\omega_i), \ i = 1, N, x_0 < \omega_1 < x_3, x_{N-2} < \omega_N < x_{N+1} \). Set \( Y = (y_n) \) where \( y_n \) is an approximation to \( y(x_n) \), \( y(x) \) being the exact solution of (1). As in [4], [8], we obtain, on neglecting the local truncation errors \( t_n \), noting \( y^{(4)} = -f(x)y + g(x) \) and \( y(x_n) = y_n \),

\[
P^2Y = -h^4DY + C, \quad P^{-1} > 0, \quad (11)
\]

(see [8]) where the tridiagonal matrix \( P = (p_{ij}) \) is given by \( p_{ij} = 2, p_{ij} = -1 \) for \( |i - j| = 1 \), otherwise \( p_{ij} = 0 \); \( D = \text{diag}(f_n) \) is a diagonal matrix and the column vector \( C \) depends on \( g(x) \) and the boundary conditions. The matrix \( P \) is symmetric and positive definite and it is known that its eigenvalues are \( 4 \sin^2(m\pi h/2), \ m \in S \). Thus the eigenvalues of \( P^2 \) are

\[
\lambda_m = 16 \sin^4(m\pi h/2), \quad m \in S. \quad (12)
\]
Lemma 5. $\pi^4h^4(1 - \pi^2h^2/6) \leq \lambda_1 \leq \pi^4h^4$.

The inequality follows from $\theta - \theta^3/6 < \sin \theta < \theta$ for $0 < \theta < \pi/2$ and $(1 - x)^n > 1 - nx$ for small values of $x$. Also the eigenvalues satisfy

$$0 < \lambda_1 < \lambda_2 < \cdots < \lambda_N.$$  \hspace{1cm} (13)

Since $P$ is symmetric, it is easy to see that

$$\|P^{-2}\|_2 = 1/\lambda_1.$$ \hspace{1cm} (14)

Lemma 6. Assume that $f(x)$ satisfies (2) and that $h_0$ is such that

$$\eta < \pi^4(1 - \pi^2h_0/6).$$ \hspace{1cm} (15)

Furthermore if $h < h_0$, and $u, v \in \Phi$ satisfy

$$P^2u = -h^4Du + C_1, \quad P^2v = -h^4Dv + C_2,$$

then $\|u - v\| \leq K(h_0)\|C_1 - C_2\|$, where

$$K(h_0) = \lambda_1^{-4}[\pi^4(1 - \pi^2h_0^2/6) - \eta]^{-1}.$$ \hspace{1cm} (16)

Proof. From the hypothesis it follows

$$P^2(u - v) = -h^4D(u - v) + (C_1 - C_2),$$

$$(u - v) = P^{-2}[-h^4D(u - v) + (C_1 - C_2)],$$

$$\|u - v\| \leq (1/\lambda_1)[\eta h^4\|u - v\| + \|C_1 - C_2\|],$$

by (2) and (14) or

$$(\lambda_1 - \eta h^4)\|u - v\| \leq \|C_1 - C_2\|.$$

Now on using Lemma 5 and (15), the result of Lemma 6 follows.

Remark. If $\eta = 0$, the constant $h_0 < 0.77$.

Lemma 7. If $f(x)$ satisfies (2) and if $Y$ is a solution of (11), then

$$\|Y\| \leq K(h_0)\|C\|.$$ \hspace{1cm} (17)

Proof. Put $u = Y$, $C_1 = C$, $v = C_2 = 0$ in Lemma 6, then (17) follows.

Theorem 2. If $f(x)$ satisfies (2), then the discrete boundary value problem (11) has a unique solution.

Proof. Clearly, Lemma 6 implies that (11) has at most one solution. Let $\Omega = \{u \in \Phi: \|u\| \leq K(h_0)\|C\|\}$. Define a mapping $T: u \rightarrow v$ by means of the relation

$$P^2v = -h^4Du + C.$$ \hspace{1cm} (18)

Since $P^{-1} > 0$, it follows that (18) has exactly one solution for a given $u$. 

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Consider $Tu = v$ and use (18) to deduce
\[
\|v\| < \left[ h^4 \|u\| + \|C\| \right] / \lambda_1
\]
\[
\leq \left[ (h^4 K(h_0) + 1) \|C\| / \left[ \pi^4 h^4 (1 - \pi^2 h_0^2 / 6) \right] \right]
\]
\[
< K(h_0) \|C\|
\]
on using (16). This proves that $T$ maps $\Omega$ into itself. Let $\epsilon > 0$ be given, we can choose $\delta(\epsilon)$
\[
\delta = \frac{\epsilon \pi^4 (1 - \pi^2 h_0^2 / 6)}{\eta}, \quad \eta \neq 0.
\]
Now if $Tu_1 = y_1, Tu_2 = y_2$, then
\[
\|Tu_1 - Tu_2\| = \|y_1 - y_2\|
\]
\[
= \left\| P^{-2}(-h^4 Du_1 + C) - P^{-2}(-h^4 Du_2 + C) \right\|
\]
\[
< h^4 \|u_1 - u_2\| / \lambda_1 < \epsilon,
\]
provided $\|u_1 - u_2\| < \delta$ given by (19) and $\lambda_1 > \pi^4 h^4 (1 - \pi^2 h_0^2 / 6)$. This shows that $T$ is continuous on $\Omega$. Hence, by Brouwer’s fixed point theorem [1], there is a $u \in \Omega$ such that $Tu = u$, and this is clearly a solution of (18) and hence of (11). This completes the proof of the theorem.

**Note.** For $\eta = 0$, an obvious modification of the argument still proves the preceding theorem.

### 4. An approximation theorem

In this concluding section we establish an a posteriori bound. We note that the system of linear equations based on (10) can be written as
\[
P^2 \bar{y} = -h^4 D\bar{y} + C + T
\]
where $\bar{y} = (\bar{y}(x_n)) \in \Phi$ and clearly
\[
\|T\| \leq \frac{1}{6} h^6 M_6
\]
where $M_6 = \max_x |d^6 y / dx^6|, \quad 0 < x < 1$. If we subtract (11) from (20), we obtain an error equation, namely
\[
P^2 E = -h^4 DE + T
\]
where $E = (e_n) \in \Phi$ and $e_n = y(x_n) - y_n$.

**Theorem 3.** If $f(x)$ satisfies (2), then for $h < h_0$: \[\|E\| = O(h^2).\]

**Proof.** From Lemma 7, it follows that
\[
\|E\| \leq K(h_0) \|T\| = O(h^2)
\]
using (16) and (22). In fact
\[
\|E\| \leq \frac{1}{6} M_6 h^2 \left[ \pi^4 (1 - \pi^2 h_0^2 / 6) - \eta \right]^{-1}
\]

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