L² HARMONIC FORMS ON ROTATIONALLY SYMMETRIC RIEMANNIAN MANIFOLDS

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ABSTRACT. The paper contains a vanishing theorem for L² harmonic forms on complete rotationally symmetric Riemannian manifolds. This theorem requires no assumptions on curvature.

This paper gives necessary and sufficient conditions for existence of L² harmonic forms on a special class of Riemannian manifolds. Manifolds of this class were called models by Greene and Wu and played a crucial part in the study of function theory on open manifolds [GW]. Throughout the paper M will denote a model of dimension n, i.e. a C∞ Riemannian manifold such that:

(1) there exists a point o ∈ M for which the exponential mapping is a diffeomorphism of T_oM onto M;

(2) every linear isometry φ: T_oM → T_oM is realized as the differential of an isometry Φ: M → M, i.e., Φ(o) = o and Φ∗(o) = φ.

Clearly, M is complete and can be identified with T_oM via exp_o. In terms of geodesic polar coordinates (r, θ) ∈ (0, ∞) × S^n−1 ≅ M\{o} the Riemannian metric ds² of M can be written as

\[ ds^2 = dr^2 + f(r)^2 d\theta^2, \]

where dθ² denotes the standard metric on S^n−1 and the function f(r) is C∞ on [0, ∞) and satisfies

\[ f(0) = 0, \quad f'(0) = 1, \quad f(r) > 0 \quad \text{for } r > 0 \]

(cf. [S, pp. 179–183]).

Complete description of the spaces \( \mathcal{H}(M) \) of L² harmonic forms is contained in the following

THEOREM. Let M be a model of dimension n > 2. Then

(i) \( \mathcal{H}^q(M) = \{0\} \) for \( q \neq 0, n/2, n \),

(ii) \( \mathcal{H}^q(M) \cong \mathcal{H}^p(M) \cong \begin{cases} \{0\} & \text{if } \int_0^\infty f(r)^{n-1} dr = \infty, \\ \mathbb{R} & \text{if } \int_0^\infty f(r)^{n-1} dr < \infty, \end{cases} \)

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(iii) \( H^k(M) = \{0\} \) if \( n = 2k \) and \( \int_1^\infty \frac{ds}{f(s)} = \infty \),

\( H^k(M) \) is a Hilbert space of infinite dimension if \( n = 2k \) and

\[
\int_1^\infty \frac{ds}{f(s)} < \infty.
\]

**Remark.** The integral in (ii) is a multiple of the volume of \( M \). Finiteness of the integral \( \int_1^\infty ds/f(s) \) implies that \( M \) is conformally equivalent to an open ball in \( \mathbb{R}^n \). If \( \int_1^\infty ds/f(s) = \infty \) then \( M \) is conformal to \( \mathbb{R}^n \).

My interest in \( L^2 \) harmonic forms is motivated in part by the well known conjecture (cf. [C, p. 44]).

**Conjecture 1.** Let \( N \) be a compact Riemannian manifold of dimension \( 2k \). If the sectional curvature of \( N \) is nonpositive the Euler characteristic \( \chi(N) \) satisfies

\[
(-1)^k \chi(N) > 0.
\]

I. M. Singer suggested that in view of the \( L^2 \) index theorem [A] an appropriate vanishing theorem for \( L^2 \) harmonic forms on the universal covering of \( N \) would imply the conjecture. To see what sort of vanishing theorem to expect, I carried out an explicit computation in the case of constant negative curvature. It turned out that the same computation yielded a more general result which is the subject of this paper. The result itself is rather surprising since the curvature of \( M \) has no effect on existence of \( L^2 \) harmonic forms of degree \( q \neq 0, n/2, n \). The vanishing in this range is a consequence of duality between forms of degree \( q \) and \( n - q \). The general question of existence of nontrivial \( L^2 \) harmonic forms on open manifolds is a very difficult one. Nevertheless, I propose hesitantly the following:

**Conjecture 2.** Let \( M \) be a simply connected complete Riemannian manifold of dimension \( n \) and of nonpositive sectional curvature. Then there are no nonzero \( L^2 \) harmonic forms on \( M \) of degree \( q \neq n/2 \).

Conjecture 2 combined with the \( L^2 \) index theorem implies Conjecture 1. Indeed, the \( L^2 \) index theorem, applied to the operator \( d + \delta \) whose index is equal to the Euler characteristic, states that \( L^2 \) harmonic forms on the universal covering \( \tilde{N} \) of \( N \) can be used to reckon the Euler characteristic of \( N \). More precisely \( \chi(N) \) is equal to the alternating sum

\[
\sum_{p=1}^{2k} (-1)^p \dim_{\pi_1(N)} H^p(\tilde{N}),
\]

where \( \dim_{\pi_1(N)} H^p(\tilde{N}) \) is the normalized dimension of \( H^p(\tilde{N}) \) with respect to the natural action of \( \pi_1(N) \) on \( H^p(\tilde{N}) \) (cf. [A]). Thus, if \( H^p(\tilde{N}) = \{0\} \) for \( p \neq k \),

\[
(-1)^k \chi(N) = \dim_{\pi_1(N)} H^k(\tilde{N}) > 0.
\]

The following example due to E. Calabi shows that one cannot expect to have \( H^p(M) = \{0\} \) for \( q \neq 0, n/2, n \) for every manifold \( M \) satisfying (1). Let \( (M_i, dr_i^2 + f_i(r_i)^2 \ d\theta_i^2) \) be a model of dimension \( n_i, i = 1, 2 \). Suppose that \( n_2 \) is
even,
\[ \int_0^\infty f(t)^{n_1-1} \, dt < \infty, \quad \int_1^\infty \frac{ds}{f_2(s)} < \infty. \]

Then, according to the theorem \( \mathcal{H}'(M) \neq \{0\} \) for \( q = 0, n_1, \mathcal{H}'(M_2) \neq \{0\} \) when \( q = n_2/2 \). The Fubini theorem and the identity
\[ \Delta_{M_1 \times M_2} = \Delta_{M_1} \otimes I + I \otimes \Delta_{M_2}, \]
\( q = n_2/2, n_2/2 + n_1 \).

The above construction cannot be used to produce a counterexample to Conjecture 2. In order that \( M_1 \times M_2 \), when equipped with the product metric, have nonpositive sectional curvature, \( M_1 \) and \( M_2 \) must have the same property. This would force the integral \( \int_0^\infty f(t)^{n_1-1} \, dt \) to diverge since the volume of complete, simply connected Riemannian manifold of nonpositive sectional curvature is infinite.

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**Proof of theorem.** According to a theorem of Andreotti and Vesentini (cf. [dR, Theorem 26]) an \( L^2 \) form \( \omega \) on \( M \) is harmonic if and only if it is closed and coclosed. Thus a \( C^\infty \) \( q \)-form \( u \) is in \( \mathcal{H}'(M) \) if and only if
\[ \int_M u \wedge \ast \omega < \infty, \quad du = 0, \quad d\ast u = 0, \]
where \( \ast \) denotes the duality operator between forms of degree \( q \) and \( n - q \). Since \( \ast \omega \wedge \ast (\ast \omega) = \omega \wedge \ast \omega \) for every form \( \omega \), \( \ast \) establishes an isomorphism between \( \mathcal{H}'(M) \) and \( \mathcal{H}^{n-q}(M) \). Let \( dV \) denote the volume element of the Riemannian metric of \( M \), and let \( \langle \cdot , \cdot \rangle \) and \( | \cdot | \) be the pointwise inner product and norm, respectively, of differential forms on \( M \). The global (integrated) inner product and norm are given by
\[ (\omega, \eta) = \int_M \omega \wedge \ast \eta = \int_M \langle \omega, \eta \rangle \, dV, \]
\[ |\omega|^2 = \int_M \omega \wedge \ast \omega = \int_M |\omega|^2 \, dV, \]
where \( \omega \) and \( \eta \) are two forms of equal degrees. Corresponding objects on \( S^{n-1} \) equipped with the standard metric will have to be considered. They will be denoted by the same symbols as their counterparts on \( M \) with a subscript 0. For example, the volume elements \( dV \) and \( dV_0 \) of \( M \) and \( S^{n-1} \), respectively, are related by \( dV = f(r)^{n-1} \, dV_0 \wedge dr \).

The case (ii) of the theorem is now trivial. If \( \omega \) is an \( L^2 \) harmonic function \( dw = 0 \) by (5), i.e., \( \omega \) is constant. Constants are in \( L^2 \) if and only if the total volume of \( M \) is finite, which gives (ii). To study the remaining cases one writes the conditions (5) in terms of geodesic polar coordinates \( (r, \theta) \). If \( \omega \) is a \( C^\infty \) \( q \)-form on \( M \setminus \{ o \} \) of degree \( q \neq 0, n \), then
\[ \omega = a(r, \theta) \wedge dr + b(r, \theta), \]
where \(a(r, \theta), b(r, \theta)\) are smooth forms on \(S^{n-1}\), depending on a parameter \(r > 0\), of degree \(q - 1\) and \(q\), respectively. Formally \(a = (-1)^{q-1} i(\partial/\partial r)\omega\), \(b = \omega - a \wedge dr\), where \(i(\partial/\partial r)\) is the interior product with the radial vector field \(\partial/\partial r\). Of course, \(a\) and \(b\) can be also regarded as forms on \(M \setminus \{o\}\).

In terms of decomposition (6) \(\ast \omega\) can be computed as follows:

\[
\ast \omega = (-1)^{n-p} f^{n-2q+1} \ast_0 a + f^{n-2q-1} b \ast_0 dr.
\]

(7)

To prove this formula one uses the fact that \(\ast\) consists essentially of taking orthogonal complement together with the identity

\[
\ast g = \ast_{**}
\]

relating duality operators on \(q\)-forms for two conformal metrics \(g\) and \(\lambda^2 g\).

Using (5), (6) and (7) one concludes that for \(\omega \in \mathcal{K}^q(M)\) the following conditions hold:

\[
\int_0^\infty \int_{S^{n-1}} (f^{n-2q+1} |a|_{\delta_0}^2 + f^{n-2q-1} |b|_{\delta_0}^2) dV_0 dr < \infty,
\]

(9)

\[
d_0 b = 0, \quad d_0 \ast_0 a = 0,
\]

\[
d_0 a + (-1)^q \frac{\partial b}{\partial r} = 0,
\]

Moreover the pointwise norm \(|\omega|^2\) is bounded near \(r = 0\), i.e.,

\[
|\omega|^2 = f^{-2(q-1)}|a|_{\delta_0}^2 + f^{-2q}|b|_{\delta_0}^2 < C \quad \text{for} \quad r \in (0, 1].
\]

Apply \(\ast_0\) to the last equation in (9) and use commutativity

\[
\frac{\partial}{\partial r} \ast_0 = \ast_0 \frac{\partial}{\partial r}
\]

to obtain the following set of conditions satisfied by \(\omega = a \wedge dr + b \in \mathcal{K}^q(M)\) on \(M \setminus \{o\}\)

\[
(a) \quad d_0 b = 0,
\]

\[
(b) \quad d_0 \ast_0 a = 0,
\]

\[
(c) \quad d_0 a + (-1)^q \frac{\partial b}{\partial r} = 0,
\]

\[
(d) \quad \frac{\partial}{\partial r} (f^{n-2q+1} a) + (-1)^q f^{n-2q-1} \delta_0 b = 0,
\]

\[
(e) \quad f^{-2(q-1)}|a|^2 + f^{-2q}|b|^2 < C \quad \text{for} \quad r \in (0, 1],
\]

\[
(f) \quad \int_0^\infty \int_{S^{n-1}} (f^{n-2q+1} |a|_{\delta_0}^2 + f^{n-2q-1} |b|_{\delta_0}^2) dV_0 dr < \infty,
\]

where \(\delta_0\) is the formal adjoint of \(d_0\) on \(S^{n-1}\). Observe now that if \(\omega \in \mathcal{K}^q(M)\) and \(b \equiv 0\), then \(a \equiv 0\). Indeed, if \(b \equiv 0\), then, by (10b) and (10c), \(a(r, \theta)\) is a harmonic form on \(S^{n-1}\) for every fixed \(r > 0\). Since \(0 < \deg a < n - 2\),

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$a(r, \theta)$ can be nonzero only if $q - 1 = \deg a = 0$, in which case $a(r, \theta)$ is independent of $\theta$. On the other hand, by (10d),

$$\frac{\partial}{\partial r}(f^{n-1}a) = 0,$$

i.e., $a = C_1 f^{-(n-1)}$ which blows up at $r = 0$ contradicting (10e) unless $C_1 = 0$.

Now eliminate $a(r, \theta)$ from the system consisting of equations (10c) and (10d). Thus apply $d_0$ to (10d) and use commutativity $d_0 \partial / \partial r = (\partial / \partial r) d_0$ to obtain

$$f^{n-2q-1}d_0b = \frac{\partial}{\partial r}(f^{n-2q+1} \frac{\partial b}{\partial r}).$$

Take the inner product (over $S^{n-1}$) of both sides of this equation with $b$ keeping $r > 0$ fixed to see that

$$\left( \frac{\partial}{\partial r}(f^{n-2q+1} \frac{\partial b}{\partial r}), b \right)_0 = f^{n-2q-1}(d_0b, d_0b)_0 > 0.$$

Therefore

$$\frac{d}{dr}(f^{n-2q+1} \frac{\partial b}{\partial r}, b)_0 = \left( \frac{\partial}{\partial r}(f^{n-2q+1} \frac{\partial b}{\partial r}), b \right)_0 + f^{n-2q+1}(\frac{\partial b}{\partial r}, \frac{\partial b}{\partial r})_0 > 0.$$

By (10e) and (4) $|b|_2^2 = O(r^2)$ for small $r$. Hence

$$\left(f^{n-2q+1} \frac{\partial b}{\partial r}, b \right)_0 = O(r^n).$$

It follows that

$$\frac{d}{dr}(b, b)_0 = 2\left( \frac{\partial b}{\partial r}, b \right)_0 > 0$$

for all $r > 0$, i.e. $\|b\|_0^2$ is a nondecreasing function of $r$. Now suppose $b \neq 0$. Since $\|b\|_0^2$ is monotone and

$$\infty > \|\omega\|^2 > \|b\|^2 = \int_0^\infty f^{n-2q-1} \|b\|_0^2 dr,$$

the integral $\int_1^\infty f^{n-2q-1} dr$ is finite. Thus for $q \neq 0, n$, $\mathcal{H}(M) \neq \{0\}$ implies that $\int_1^\infty f^{n-2q-1} dr$ is finite. By duality $\mathcal{H}(M) \cong \mathcal{H}^{-n}(M)$, i.e. if $\mathcal{H}(M) \neq \{0\}$, then the two integrals $\int f^{n-2q-1} dr$, $\int f^{n+2q-1} dr$ are simultaneously finite. If $n = 2q$ the two integrands are the same. If, on the other hand, $n - 2q \neq 0$ then $(n - 2q - 1)(-n + 2q - 1) = 1 - (n - 2q)^2$. Thus either one of the exponents is equal to zero, or the two exponents have opposite signs. In both cases one of the integrals has to diverge, which proves that $\mathcal{H}(M) = \{0\}$ if $q \neq 0, n/2, n$. This still leaves the possibility that, for $n = 2k$, $\mathcal{H}(M) \neq \{0\}$ provided $f_1^\infty f^{-1} dr < \infty$. Such is the case and, in fact, $\mathcal{H}(M)$ has infinite dimension. The last assertion will follow from the following:
Lemma. Let $M$ be a model with the metric $ds^2 = dr^2 + f(r)^2 \, d\theta^2$. Define

$$R(r) = e^{\int \frac{dr}{f(r)}}.$$

Then the mapping $F: M \setminus \{o\} \to \mathbb{R}^n \setminus \{o\}$ given (in terms of polar geodesic coordinates $(r, \theta)$ on $M$ and polar coordinates on $\mathbb{R}^n$) by $F(r, \theta) = (R, \theta)$ extends to a $C^1$ conformal diffeomorphism of $M$ onto an open ball of (possible infinite) radius equal to $\int \frac{ds}{f(s)}$ centered at the origin. Moreover, $F$ is $C^\infty$ on $M \setminus \{o\}$.

Remark. The lemma is due to Milnor [M] for $n = 2$, in which case $F$ is $C^\infty$ everywhere. The proof for $n > 2$ is essentially the same and will not be repeated here. If $n > 2$, the restriction of $F$ to every plane through $o$ is $C^\infty$. It is likely that $F$ is $C^\infty$, but the regularity asserted in the lemma is sufficient for the purpose at hand.

To finish the proof of the theorem assume the lemma and suppose $\dim M = 2k$. By (8) the $\bullet$ operator acting on forms of degree $k$ depends only on the conformal structure. Thus all conditions in (5) are conformally invariant. Assume that $\int \frac{ds}{f(s)} < \infty$ and let $B$ be the open ball in $\mathbb{R}^n$ of radius $\int \frac{ds}{f(s)}$. The space of all $C^\infty$ $k$-forms $\eta$ on $\mathbb{R}^n$ which satisfy the equations $d\eta = 0$, $d^\bullet \eta = 0$ ($\bullet$ induced by the standard flat metric) has infinite dimension (e.g., if $h(y_1, y_2, \ldots, y_{k+1})$ is a nonconstant harmonic function on $\mathbb{R}^{k+1}$, then

$$\eta = d(h(x_1, x_{k+1}, \ldots, x_n) \, dx_1 \ldots dx_{k-1})$$

satisfies the two equations). Restrictions of such forms to $B$ are clearly in $L^2$. Thus the space $\mathcal{K}$ of $k$-forms on $B$ satisfying conditions (5) with respect to the flat metric has infinite dimension. By the lemma and the conformal invariance, the space $F^* \mathcal{K}$ consists of forms $\omega$ of degree $k$ which are continuous, square integrable on $M$, $C^\infty$ on $M \setminus \{o\}$ and satisfy $d\omega = d^\bullet \omega = 0$ on $M \setminus \{o\}$. Standard regularity theorem shows that every $\omega \in F^* \mathcal{K}$ is in fact $C^\infty$ and harmonic at every point of $M$. It follows that $F^*$ establishes an isomorphism between $\mathcal{K}$ and $\mathcal{K}^k(M)$ and that $\mathcal{K}^k(M)$ has infinite dimension.

References


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