THE GENERATION OF NONLINEAR EQUIVARIANT DIFFERENTIAL OPERATORS

ROBERT DELVER

ABSTRACT. Finite generation results are given for the set of smooth nonlinear differential operators: \( C^\infty(M, N) \to C^\infty(M, R) \) of order \( < k \) which are equivariant with respect to the action of a Lie group on the base manifold \( M \).

1. Introduction. Let \( G \) be a Lie group acting by diffeomorphisms \( \phi_g, g \in G \), on a smooth manifold \( M \), \( N \) a smooth manifold and let \( \mathcal{D}_k, k \in \{ \infty, 1, 2, 3, \ldots \} \), denote the real vector space of smooth nonlinear differential operators of order \( < k \) of \( C^\infty(M, N) \) into \( C^\infty(M, R) \). The action of \( G \) on \( M \) lifts to \( C^\infty(M, N) \) by \( g \cdot f = f \circ \phi_g^{-1}, f \in C^\infty(M, N), g \in G \). Let \( \mathcal{D}_k^G \) denote the \( G \)-equivariant elements of \( \mathcal{D}_k \). Full definitions are given in \( \S 2 \).

There are two equivariance preserving structures on \( \mathcal{D}_\infty \) each with its own generation problem. The first structure is a multiplication: \( \mathcal{D}_k \times \mathcal{D}_k \to \mathcal{D}_k \), defined by

\[
\mathcal{F}_1 \cdot \mathcal{F}_2(f) = \mathcal{F}_1(f)\mathcal{F}_2(f), \quad f \in C^\infty(M, N). \tag{1.1}
\]

If \( N = R \), a second structure is induced by the composition \( \mathcal{D}_{k_1} \times \mathcal{D}_{k_2} \to \mathcal{D}_{k_1 + k_2} \) given by

\[
\mathcal{F}_1 \mathcal{F}_2(f) = \mathcal{F}_1(\mathcal{F}_2(f)), \quad f \in C^\infty(M, R). \tag{1.2}
\]

The main results of this paper are two finiteness theorems, one for each of these structures.

THEOREM 1. Let \( G \) be a compact Lie group, \( M \) a smooth \( G \)-manifold of finite orbit type and \( N \) a smooth manifold then, for each \( k \in \{ 0, 1, 2, \ldots \} \), there exist \( \mathcal{Q}_1, \ldots, \mathcal{Q}_i \in \mathcal{D}_k^G \) such that \( \mathcal{F} \in \mathcal{D}_k^G \) iff \( \mathcal{F} = f(\mathcal{Q}_1, \ldots, \mathcal{Q}_i) \), for some \( f \in C^\infty(R^i) \).

This theorem is based on a theorem of Schwarz [10], the proof is given in \( \S 2 \).

Let \( C^\infty(M)^G \) denote the \( G \)-invariant elements of \( C^\infty(M) \). A function \( \xi: M \to R^l \) is called a finite generator for \( C^\infty(M)^G \) iff \( C^\infty(M)^G = \xi^* C^\infty(M) \).

Received by the editors December 5, 1977 and, in revised form, February 15, 1978 and October 4, 1978.


Key words and phrases. Equivariant differential operators, transformation groups, differential invariants.

1 The preparation of this paper was supported by the National Research Council under Grant A8731.

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0002-9939/79/0000-0570/$03.00

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We let $\Gamma^G(T(M))$ denote the $C^\infty(M)^G$-module of invariant vector fields on $M$. In the case where $M$ is a principal $G$-bundle finite generators exist both for $C^\infty(M)^G$ and for the module $\Gamma^G(T(M))$ (Lemmas (3.1) and (3.2)). Moreover, \{$X_1, \ldots, X_n$\} $\subset \Gamma^G(T(M))$ is a generator for $\Gamma^G(T(M))$ iff \{$X_1(x), \ldots, X_n(x)$\} generates the vector space $T_x(M)$ for all $x \in M$ (Lemma (3.3)). Let $N = \mathbb{R}$.

**Theorem 2.** Let $M$ be a principal $G$-bundle with fibration \{$M, \pi, B$\}, $\xi : M \rightarrow \mathbb{R}^l$ a generator for $C^\infty(M)^G$ and \{$X^1, \ldots, X^n$\} a generator for $\Gamma^G(T(M))$. Then

$$\mathfrak{G}_k(M)^G = \{\xi, (X^\alpha)_{|\alpha| < k}\}^* C^\infty(\mathbb{R}^l \times \mathbb{R}^n),$$

for $k > 1$.

$(X^\alpha)_{|\alpha| < k}$ denotes the sequence of all $X^\alpha$ with $|\alpha| < k$ in lexicographical order and \{$\xi, (X^\alpha)_{|\alpha| < k}\}^* C^\infty(\mathbb{R}^l \times \mathbb{R}^n)$ is the set of operators of the form $a(\xi, (X^\alpha)_{|\alpha| < k}), a \in C^\infty(\mathbb{R}^l \times \mathbb{R}^n)$.

In a somewhat different context, problems of this type were studied by Lie [8], by Tresse [11] and more recently by Kumpera [7].

Thanks are due to Ivan Kupka for some helpful discussion.

**2. The verification of Theorem 1.** Let $J^k(M, N)$ be the $k$th jet bundle from $M$ into $N$ with source map $\alpha$ and target map $\beta$. If $P$ and $Q$ are smooth manifolds, $\mu : P \rightarrow M$ a diffeomorphism, $\nu : N \rightarrow Q$ a smooth map, then $(J^k\mu)^* : J^k(M, N) \rightarrow J^k(P, N)$ and $(J^k\nu)_* : J^k(M, N) \rightarrow J^k(M, Q)$ are defined by

$$(J^k\mu)^*(\sigma) = j^k_{\mu^{-1}(\alpha(\sigma))} f \circ \mu \quad \text{and} \quad (J^k\nu)_*(\sigma) = j^k_{\nu(\alpha(\sigma))} \circ f,$$

where $f$ represents $\sigma$.

The action of $G$ on $M$ lifts to a smooth action on $J^k(M, N)$ by

$$(g \cdot \sigma) = (j^k_{g^{-1}} \circ \sigma). \quad (2.1)$$

Let $\pi_k$ be the canonical projection of $J^{k+1}(M, N)$ onto $J^k(M, N)$ and put

$$D_k = C^\infty(J^k(M, N), \mathbb{R}).$$

$D_\infty$ is the inductive limit as $k \rightarrow \infty$ of $(D_k, \pi_k^*)$,

where $\pi_k^*$ is the map from $D_k$ to $D_{k+1}$ given by $\pi_k^* F = F \circ \pi_k$.

The set of smooth nonlinear differential operators from $C^\infty(M, N)$ into $C^\infty(M, \mathbb{R})$ of order $< k$, $k \in \{\infty, 1, 2, 3, \ldots\}$ is denoted by $\mathfrak{D}_k \subseteq \mathfrak{D}_k^G$. In this case $F$ is called the symbol of $\mathfrak{F}$ or $F = \text{sym } \mathfrak{F}$. The $G$-equivariant elements of $\mathfrak{D}_k$ are denoted by $\mathfrak{D}_k^G$, the $G$-invariant elements of $D_k$ by $D_k^G$.

**Proposition.** $\mathfrak{F} \in \mathfrak{D}_k^G$ iff $\text{sym } \mathfrak{F} \in D_k^G$.

**Proof.** Let $F$ be the symbol of $\mathfrak{F}$. If $F$ is $G$-invariant, $f \in C^\infty(M, N)$, $x \in M$, then

$$\mathfrak{F}(g \cdot f)(x) = F(j^k_{g^{-1}} \circ f) = (g^{-1} \cdot F)(j^k_{g^{-1}} \circ f) = F(j^k_{g^{-1}} \circ f)$$

$$= (\mathfrak{F}f)(g^{-1} \cdot x) = (g \cdot \mathfrak{F}f)(x).$$
Conversely, if $\mathcal{T}$ is $G$-equivariant and $\sigma \in J^k(M, N)$ with $a(\sigma) = x$ is represented by $f$ then
\[ g \cdot F(\sigma) = F(j^k_{g^{-1}}g^{-1} \cdot f) = \mathcal{T}(g^{-1} \cdot f)(g^{-1} \cdot x) = (g^{-1} \cdot \mathcal{T}f)(g^{-1} \cdot x) = \mathcal{T}f(x) = F(a). \]

(2.3) **Lemma.** Let $G$ be a compact Lie group and $M$ a smooth $G$-manifold with orbit structure of finite type (see [9]), then the induced action on $J^k(M, N)$ is of finite orbit type as well.

**Proof.** If $N = \mathbb{R}^n$ and $M$ is a linear $G$-space then $J^k(M, N)$ is a linear $G$-space which is of finite orbit type. In the general case we can assume by the Whitney and Mostov embedding theorems that $N$ is smoothly embedded in $\mathbb{R}^n$ and $M$ is smoothly equivariantly embedded in a Euclidean $G$-space $\mathbb{R}^m$. It will suffice to show that $J^k(M, N)$ is equivariantly embedded in $J^k(\mathbb{R}^m, \mathbb{R}^n)$. Let $\pi: Z \to M$ be an equivariant tubular neighbourhood of $M$ in $\mathbb{R}^m$. Since $Z$ is an open $G$-invariant set in $\mathbb{R}^m$, $J^k(Z, \mathbb{R}^n)$ is an open $G$-submanifold of $J^k(\mathbb{R}^m, \mathbb{R}^n)$, so we need only show that $J^k(M, N)$ is equivariantly embedded in $J^k(Z, \mathbb{R}^n)$.

Let $i$ be the inclusion map of $N$ in $\mathbb{R}^n$. Clearly, $(J^k i)_*: J^k(M, N) \to J^k(M, \mathbb{R}^n)$ and $(J^k \pi)_*: J^k(M, \mathbb{R}^n) \to J^k(Z, \mathbb{R}^n)$ are equivariant embeddings. $(J^k \pi)_* \circ (J^k i)_*$ is the desired equivariant embedding of $J^k(M, N)$ into $J^k(Z, \mathbb{R}^n)$. □

Thanks are due to the referee of an earlier version of this section for shortening my original proof.

By Lemma (2.3), the conditions of Theorem 1 imply that the $G$-manifold $J^k(M, N)$ is of finite orbit type. By a theorem of G. W. Schwarz [10, Theorem 2], there exist $A_1, \ldots, A_i \in D^G_k$, such that $F \in D^G_k$ iff $F = f(A_1, \ldots, A_i)$, for some $f \in C^\infty(\mathbb{R}^l, \mathbb{R})$. Hence, the operators $\partial_l, 1 < l < i$, may be chosen as those with $\text{sym } \partial_l = A_l$.

3. **The verification of Theorem 2.** In this section, $M$ is a smooth principal $G$-bundle with corresponding fibration $(M, \pi, B)$. $m = \dim M, a = \dim G$ and $b = m - a = \dim B$.

(3.1) **Lemma.** There exists an invariant generator $\xi: M \to \mathbb{R}^l, l \leq 2b + 1$, for $C^\infty(M)^G$.

**Proof.** By Whitney's embedding theorem there exists an embedding $\zeta$ of $B$ into $\mathbb{R}^p, p \leq 2b + 1$. Since $C^\infty(M)^G = \pi^*C^\infty(B)$, we may choose $\xi = \zeta \circ \pi$. □

The action of $G$ on $M$ lifts in the usual way to $T(M)$ by $g \cdot (x, v) = (g \cdot x, g \cdot v)$, where $x \in M, v \in T_x(M), g \cdot x = \phi(x)$ and $g \cdot v = d\phi_g(x)v$. As in §1, $\Gamma^G(T(M))$ denotes the $C^\infty(M)^G$-module of $G$-invariant vector fields on $M$.

(3.2) **Lemma.** There exists a finite generator for $\Gamma^G(T(M))$. 

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ROBERT DELVER

PROOF. For each \( x \in M \), let \( H_x(M) \) be the horizontal tangent space to \( M \) at \( x \) with respect to a given principal connection \( \mathcal{D} \) on \( M \) and let \( V_x(M) \) be the vertical tangent space at \( x \). In \( T(M) \) we consider the subbundles \( H(M) = \bigcup_{x \in M} H_x(M) \) and \( V(M) = \bigcup_{x \in M} V_x(M) \). Since \( T(M) = H(M) \oplus V(M) \), we need only show that finite generators exist for \( \Gamma^G(H(M)) \) and \( \Gamma^G(V(M)) \), the \( C^\infty(M)^G \)-modules of \( G \)-invariant horizontal and vertical vector fields on \( M \).

First we construct a finite generator for \( \Gamma^G(H(M)) \). By Whitney's embedding theorem we may assume that \( B \) is embedded in \( \mathbb{R}^p, p < 2b + 1 \). Projecting the canonical basis of \( T(\mathbb{R}^p) \) onto \( T(B) \) we obtain a generator for \( T(B) \). The horizontal liftings with respect to \( \mathcal{D} \) of the elements of this generator constitute a generator for \( \Gamma^G(H(M)) \).

\( V(M) \) is a \( G \)-subbundle of the \( G \)-vector bundle \( T(M) \). The action of \( G \) on \( V(A_f) \) is given by

\[
(g \cdot (x, v)) = (g \cdot x, g \cdot v), \quad (3.3)
\]

where \((x, v) \in M \times V_x(M)\).

Let \( \text{Lie}(G) \) be the Lie algebra of \( G \). For \( l \in \text{Lie}(G) \), let the vertical vector field \( \tilde{l} \) on \( M \) be defined by

\[
\tilde{l}(x) = \left. \frac{d}{dt} (e^{t l}(x)) \right|_{t=0}, \quad x \in M. \quad (3.4)
\]

We define a left \( G \)-action on \( M \times \text{Lie}(G) \) by

\[
g \cdot (x, l) = (g \cdot x, \text{Ad}(g) l) \quad (3.5)
\]

where \( \text{Ad}(g) \) is the adjoint action of \( g \in G \) on \( \text{Lie}(G) \). With this action, \( M \times \text{Lie}(G) \) is isomorphic, as a \( G \)-vector bundle, to \( V(M) \). An isomorphism is given by \( \omega: M \times \text{Lie}(G) \to V(M): \)

\[
\omega(x, l) = (x, \tilde{l}(x)). \quad (3.6)
\]

We check that \( \omega \) is \( G \)-equivariant:

\[
\omega(g \cdot (x, l)) = \omega(g \cdot x, \text{Ad}(g) l) = (g \cdot x, \overline{\text{Ad}(g)} \tilde{l}(g \cdot x)) = (g \cdot x, g \cdot \tilde{l}(x)) = g \cdot \omega(x, l),
\]

for all \( g \in G \) and \((x, l) \in M \times \text{Lie}(G)\).

Let \( E = M \times^G \text{Lie}(G) \) be the vector bundle over \( B \) of fiber type \( \text{Lie}(G) \) associated with the principal bundle \( M \) and the adjoint action of \( G \) on \( \text{Lie}(G) \) [4, XVI, 16.14.7]. In our case \( E \) is the quotient of \( M \times \text{Lie}(G) \) by the action defined by (3.5). The invariant vertical vector fields on \( M \) are in bijective correspondence to the sections of \( E \), (see e.g. [6, Theorem 4.8.1]). Since the \( C^\infty(B) \)-module of cross sections of \( E \) is finitely generated, [5, p. 76, Lemma 2], the same is true for the \( C^\infty(M)^G \)-module \( \Gamma^G(V(M)) \). This completes the proof.

(3.7) Lemma. \( \{X^1, \ldots, X^n\} \subset \Gamma^G(T(M)) \) is a generator for \( \Gamma^G(T(M)) \) iff \( \{X^1(x), \ldots, X^n(x)\} \) generates the vector space \( T_x(M) \), for all \( x \in M \).
Proof. Clearly a generator of $\Gamma^G(T(M))$ generates the individual tangent spaces. To prove the converse, let

$$\{X^1, \ldots, X^n \} \subset \Gamma^G(T(M)) \quad (3.8)$$

be such $\{X^1(x), \ldots, X^n(x)\}$ generates $T_x(M)$, for all $x \in M$. For each $x \in X$ we may choose a subset of (3.8)

$$\{X^{i(x)}, \ldots, X^{n(x)}\} \quad (3.9)$$

which, evaluated at $x$, is a basis for $T_x(M)$. Being a basis is an open condition so there exists an open neighbourhood $0_x$ of $x$ such that (3.9), evaluated at any $y \in 0_x$ is a basis for $T_y(M)$. Because $g \cdot X^i(x) = X^i(g \cdot x)$, $x \in M$ and $1 < i < n$, and since the action of $G$ on $T(M)$ preserves linear independence (3.9), evaluated at $y$ is a basis for $T_y(M)$ for all $y \in G \cdot 0_x = \pi^{-1}(U_x)$, where $U_x = \pi(0_x)$.

Let $\{V_a\}_{a \in I}$ be a locally finite refinement of the covering $\{U_x\}_{x \in M}$ of $B$. By the above construction, for each $\alpha \in I$, we have a subset of $m$ elements of (3.8).

$$\{X^{i_1(a)}, \ldots, X^{i_m(a)}\}, \quad (3.10)$$

which, evaluated at any $x \in \pi^{-1}(V_a)$, is a basis for $T_x(M)$. Hence any $Y \in \Gamma^G(T(M))$ is of the form

$$Y(x) = \sum_{i=1}^{m} a_{i(a)}(x) X^{i(a)}(x), \quad x \in \pi^{-1}(V_a), \quad (3.11)$$

where $a_{i(a)} \in C^\infty(\pi^{-1}(V_a))^G$, $1 < i < m$, $\alpha \in I$.

Let $\{f_a\}_{a \in I}$ be a partition of unity subordinate to $\{V_a\}_{a \in I}$ with supp $f_a \subset V_a$, $\forall \alpha \in I$. Then

$$Y = \sum_{\alpha \in I} f_a \circ \pi \sum_{i=1}^{m} a_{i(a)} X^{i(a)} \quad (3.10)$$

which may be written as

$$Y = \sum_{i=1}^{n} b_i X^i, \quad (3.11)$$

with $b_i \in C^\infty(M)^G$, $1 < i < m$, since each $b_i$ is locally a finite sum of functions $(f_a \circ \pi) a_{i(a)}$ which are smooth and $G$-invariant.

Proof of Theorem 2. First we consider the case where $M$ is a trivial principal bundle: $M = V \times G$. Moreover we assume that $V$ is an open set of $\mathbb{R}^b$. For $k \in \{1, 2, 3, \ldots \}$ let $A_k: J^k(V \times G) \to J^k_{V \times (e)}(V \times G)$ be defined by

$$A_k(\sigma) = g^{-1} \cdot \sigma \quad \text{if } \alpha(\sigma) = (v, g). \quad (3.12)$$

Then $\{J^k(V \times G), A_k, J^k_{V \times (e)}(V \times G)\}$ is a fibration of the principal $G$-bundle $J^k(V \times G)$, so

$$C^\infty(J^k(V \times G))^G = A_k^* C^\infty(J^k_{V \times (e)}(V \times G)). \quad (3.13)$$
The mapping $B_k$, defined by

$$B_k(j^{k}_{(b,e)}f) = (b, j^{k}_{(0,e)}(f \circ t_b)), \quad (3.14)$$

where $f \in C^\infty(\mathbb{R}^b \times G)$ and $t_b$ is the translation in $\mathbb{R}^b \times G$, given by $t_b(a, h) = (a + b, h)$, with $b \in \mathbb{R}^b$ and $(a, h) \in \mathbb{R}^b \times G$, is a diffeomorphism of $J^{k}_{(e)}(\mathbb{R}^b \times G)$ onto $\mathbb{R}^b \times J^{k}_{(e)}(\mathbb{R}^b \times G)$. The space $J^{k}_{(e)}(\mathbb{R}^b \times G)$ carries a natural linear structure. By choosing a basis it is identified with $\mathbb{R}^N$, $N = \dim J^{k}_{(e)}(\mathbb{R}^b \times G)$ and we may consider $B_k$ as a diffeomorphism of $J^{k}_{(e)}(\mathbb{R}^b \times G)$ onto $\mathbb{R}^b \times \mathbb{R}^N$. From (3.13) we obtain

$$C^\infty(J^{k}(V \times G))^{G} = (B_k \circ A_k)^* C^\infty(V \times \mathbb{R}^N). \quad (3.15)$$

The canonical projections of $\mathbb{R}^b \times \mathbb{R}^N$ on its first and second factors are denoted by $p_1$ and $p_2$. Let $\mathcal{G}_k$: $C^\infty(V \times G) \to C^\infty(V \times G, \mathbb{R}^N)$ be the $G$-equivariant linear operator defined by

$$\mathcal{G}_k = p_2 \circ B_k \circ A_k \circ j^k \quad (3.16)$$
and let $\pi'$ be the canonical projection of $V \times G$ onto $V$. From Proposition (2.2) and formula (3.15) it follows that $\mathcal{G} \in \mathcal{G}_k^{G}(V \times G)$ iff there exists some $a \in C^\infty(\mathbb{R}^b \times \mathbb{R}^N)$ such that, for all $f \in C^\infty(V \times G)$,

$$\mathcal{G}f = a(p_1 \circ B_k \circ A_k \circ j^k, p_2 \circ B_k \circ A_k \circ j^k) \quad (3.17)$$
which equals $a(\pi', \mathcal{G}_k f) = a(\pi', \mathcal{G}_k f)$. Hence

$$\mathcal{G}^{G}_k(V \times G) = (\pi', \mathcal{G}_k)^* C^\infty(\mathbb{R}^b \times \mathbb{R}^N). \quad (3.18)$$

Let $\mathcal{G}^i_k$, $1 < i < N$, denote the $i$th component of $\mathcal{G}_k$. Clearly each $\mathcal{G}^i_k$ is a linear $G$-equivariant differential operator on $C^\infty(V \times G)$.

It is easy to see that there exists an invariant basis

$$\{ Y^1, \ldots, Y^m \} \quad (3.19)$$
for $T(V \times G)$ ($m = \dim V \times G$). From the linearity of $\mathcal{G}_k$ it follows that each $\mathcal{G}^i_k$, $1 < i < N$, can be written uniquely as

$$\mathcal{G}^i_k = \sum_{|\alpha| < k} a_i^\alpha Y^\alpha, \quad a_i^\alpha \in C^\infty(V \times G), \quad (3.20)$$
(see e.g. [12, Theorem 1.1.2]). It follows from the $G$-equivariance of $\mathcal{G}^i_k$ and of the operators $Y^\alpha$, $|\alpha| < k$, that the coefficients $a_i^\alpha$, $|\alpha| < k$, $1 < i < N$, are $G$-invariant. Thus we get the $\mathcal{G}^i_k$ in the form

$$\mathcal{G}^i_k = \sum_{|\alpha| < k} (b_i^\alpha \circ \pi') Y^\alpha, \quad 1 < i < N, \quad (3.21)$$
where $b_i^\alpha \in C^\infty(V)$, $|\alpha| < k$, $1 < i < N$.

Substituting (3.21) into (3.18) we obtain that $\mathcal{G} \in \mathcal{G}^{G}_k(V \times G)$ iff there exists some $a \in C^\infty(\mathbb{R}^b \times \mathbb{R}^N)$ such that

$$\mathcal{G}f = a(\pi', \sum_{|\alpha| < k} (b_i^\alpha \circ \pi') Y^\alpha f, \ldots, \sum_{|\alpha| < k} (b_N^\alpha \circ \pi') Y^\alpha f), \quad (3.22)$$
for all $f \in C^\infty(V \times G)$. The right-hand side of (3.22) is just a function of $\pi'$. 

and the operators $Y^\alpha$, $|\alpha| < k$. Conversely, any such function represents an element of $\mathcal{D}_k^G(V \times G)$. Hence,
\begin{equation}
\mathcal{D}_k^G(V \times G) = (\pi', (Y^\alpha)_{|\alpha| < k})C^\infty(\mathbb{R}^b \times \mathbb{R}^{m^*}).
\end{equation}

Let $(U_e)_{e \in I}$, where $I$ is some index set, be a locally finite atlas for $B$ such that the principal bundles $M_e$ induced by $M$ over $U_e$, $e \in I$, are trivializable. Then for each $e \in I$ there exists an isomorphism $\lambda_e$ of $M_e$ onto the product bundle $V_e \times G$, where $V_e \subset \mathbb{R}^b$ is the parameter domain of $U_e$.

We define a bijection $\Lambda^k_e$ of $\mathcal{D}_k^G(V_e \times G)$ onto $\mathcal{D}_k^G(M_e)$ by
\begin{equation}
(\Lambda^k_e f)(\xi) = \lambda^*_e(f \circ \lambda_e^{-1}), \quad f \in C^\infty(M_e).
\end{equation}

Let $\pi'_e$ be the canonical projection of $V_e \times G$ onto $V_e$ and let $(Y^1_e, \ldots, Y^m_e)$ be a $G$-invariant basis for $T(V_e \times G)$. From (3.23) and (3.24) we obtain that $\mathbb{B} \in \mathcal{D}_k^G(M_e)$ iff
\begin{equation}
\mathbb{B} = \Lambda^k_e(q(\pi'_e, (Y^\alpha)_{|\alpha| < k})),
\end{equation}
for some $q \in C^\infty(\mathbb{R}^l \times \mathbb{R}^{m^*})$. Or
\begin{equation}
\mathbb{B} = q(\pi'_e \circ \lambda_e, ((\Lambda^k_e Y^\alpha)_e)_{|\alpha| < k}).
\end{equation}

Let $\xi_e: M_e \rightarrow \mathbb{R}^l$ and $(X^1_e, \ldots, X^n_e)$ be the restrictions of the given generators for $C^\infty(M)^G$ and $\Gamma^G(T(M))$ to $M_e$. Then
\begin{equation}
\pi'_e \circ \lambda_e = d \circ \xi_e,
\end{equation}
for some $d \in C^\infty(\mathbb{R}^l, \mathbb{R}^b)$. Since $(X^1_e(x), \ldots, X^n_e(x))$ generates $T_x(M_e)$, $\forall x \in M_e$, it follows from Lemma (3.7) that
\begin{equation}
\Lambda^k_e Y^i_e = \sum_{j=1}^n (e^j_e \circ \xi_e)X^j_e,
\end{equation}
where $e^j_e \in C^\infty(\mathbb{R}^l)$, $1 \leq i < m$, $1 \leq j < n$. After substitution of (3.27) and (3.28) into (3.26) it is easy to see that we may write this equality as
\begin{equation}
\mathbb{B} = r(\xi_e, (X^\alpha)_e)_{|\alpha| < k},
\end{equation}
for some $r \in C^\infty(\mathbb{R}^l, \mathbb{R}^{m^*})$.

Let $(u_e)_{e \in I}$ be a partition of unity on $B$ subordinate to $(U_e)_{e \in I}$ with $\text{supp}(u_e) \subset U_e$, $e \in I$. For a given $\xi \in \mathcal{D}_k^G(M_e)$ let $\xi_e \in \mathcal{D}_k^G(M_e)$ be the restriction of $\xi$ to $C^\infty(M_e)$. ($F_e$ is defined by $\text{sym}F_e = \text{sym}F_e|J^u(M_e)$.) By (3.29) there exists a $r_e \in C^\infty(\mathbb{R}^l \times \mathbb{R}^{m^*})$ such that
\begin{equation}
F_e = r_e(\xi_e, (X^\alpha)_e)_{|\alpha| < k}.
\end{equation}

So
\begin{equation}
F = \sum_{e \in I} (u_e \circ \pi)r_e(\xi_e, (X^\alpha)_e)_{|\alpha| < k}.
\end{equation}

Since
\begin{equation}
(U_e \circ \pi)r_e(\xi_e, (X^\alpha)_e)_{|\alpha| < k}) = (U_e \circ \pi)r_e(\xi_e, (X^\alpha)_e)_{|\alpha| < k})
\end{equation}
it follows from (3.31) that
\begin{equation}
F = a(\xi, (X^\alpha)_{|\alpha| < k}),
\end{equation}
for some $a \in C^\infty(\mathbb{R}^l \times \mathbb{R}^{n^*})$. Conversely, (3.32) implies that $\mathcal{F} \in \mathcal{D}_k^G(M)$. This completes the proof.

BIBLIOGRAPHY