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AN ABSTRACT BOREL DENSITY THEOREM

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ABSTRACT. In this paper an abstract form of the Borel density theorem and related results is given centering around the notion of the author's of a (finite dimensional) "admissible" representation. A representation ρ is strongly admissible if each $\Lambda^r\rho$ is admissible. Although this notion is somewhat technical it is satisfied for certain pairs (G, ρ); e.g., if G is minimally almost periodic and ρ arbitrary, if G is complex analytic and ρ holomorphic. If G is real analytic with radical R, G/R has no compact factors and R acts under ρ with real eigenvalues, then ρ is strongly admissible. If in addition G is algebraic/$\mathbf{R}$, then each $\mathbf{R}$-rational representation is admissible. The results are proven in three stages where V is defined either over $\mathbf{R}$ or $\mathbf{C}$.

If ρ is a strongly admissible representation of G on V, then each G-invariant measure μ on $\mathcal{G}(V)$, the Grassmann space of V, has support contained in the G-fixed point set.

If ρ is a strongly admissible representation of G on V and G/H has finite volume, then each H-invariant subspace of V is G-invariant.

If G is an algebraic subgroup of $\mathrm{Gl}(V)$ and each rational representation is admissible, then H is Zariski dense in G.

The Borel density theorem [1] states that if G is a semisimple linear algebraic group/$\mathbf{R}$ and H is a discrete, or more generally a Euclidean closed subgroup such that G/H has finite volume (or more generally has property S) then the algebraic (Zariski) hull $H^{\#}$ of H equals G. In [4] I proved certain generalizations of the Borel density theorem in various forms. Principally this was done for minimally almost periodic groups (Furstenberg's case [3]), complex analytic linear groups (done independently by a different method by S. P. Wang [5]) and real analytic linear groups G with radical R and with the property that G/R has no compact factors, R acts with real eigenvalues and H is a lattice in G. While there was a certain underlying unity to these results the methods seemed, on the surface, to be ad hoc. Relying heavily on [4] we present here an abstract form of the theorem which applies to all these cases simultaneously, gives the new result contained in Theorem D, and which in addition proves a generalization of the last-named result of [4]. Finally, the results are now in a form where they could be directly applied to other situations.

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In what follows, G will be a locally compact group, ρ a continuous finite dimensional real or complex linear representation of G on V, $\mathcal{G}(V)$ denotes the Grassmann space of all subspaces of V and $G \times \mathcal{G}(V) \to \mathcal{G}(V)$ will denote the induced action from ρ on $\mathcal{G}(V)$. If $G \times X \to X$ is any action of G on a space X then X_{fix} or $X_{\text{fix},G}$ denotes the set of G-fixed points while X_c or $X_{c,G}$ denotes the G-bounded points of X; that is those with compact G-orbit closures.

An examination of the results of [4] leads to the following definition. We shall say a representation ρ of G is admissible[2] if there is a family of subgroups $\{H_i\colon i \in I\}$ which generate G and such that each restriction $\rho_i = \rho|H_i$ has the following properties.

(i) $V_{c,\rho_i,H_i} = V_{\text{fix},\rho_i,H_i}$.

(ii) H_i has no closed subgroup of finite index.

(iii) For each (ρ_i, H_i) invariant subspace W of V either ρ_i acts on W by scalars or else the function $g \mapsto \det(g|W)/\|g|W\|^{\dim W}$ vanishes at ∞ on $\rho(H_i)$, where $\|\ \|$ is any convenient Banach algebra norm on End W.

We shall say that ρ is strongly admissible if each rth-exterior power $\Lambda^r\rho$ acting on $\Lambda^r V$ is admissible for $r = 1, \ldots, \dim V$.

The importance of this notion is illustrated by the following theorem which is a slight modification of (1.11) of [4].

THEOREM A. *If ρ is a strongly admissible representation of G on V then each G invariant measure μ on $\mathcal{G}(V)$ has* $\operatorname{supp} \mu \subseteq \mathcal{G}(V)_{\text{fix}}$.

We now give sufficient conditions for representations to be admissible.

THEOREM B. *If*

(1) *G is minimally almost periodic then any continuous representation ρ is admissible,*

(2) *G is complex analytic then any holomorphic ρ is admissible.*

In particular all such representations are strongly admissible.

(3) *Suppose G is a real analytic group with G/R having no compact factors and ρ is a representation with the property that R acts with real eigenvalues then ρ is strongly admissible.*

(4) *Let G be a real linear algebraic subgroupof* $\mathrm{Gl}(V)$ *which is Euclidean connected, such that G/R has no compact factors and R acts with real eigenvalues then each* $\mathbf{R}$*-rational representation ρ is admissible.*

The first three statements were proven in [4]. To prove (4) we note that since ρ is an analytic representation: $G \to \mathrm{Gl}(W)$ we know that $\rho(R) =$ rad of $\rho(G)$ and that $\rho(G)/\rho(R)$ has no compact factors. By (3) it is sufficient to see that $\rho(R)$ acts with only real eigenvalues on W. Since R is a connected soluble algebraic group acting with real eigenvalues on V, R is simply connected by (3.2)a of [4]. But ρ is $\mathbf{R}$-rational and by Lie's theorem we may

[2]We take the liberty of using the term admissible even though it is used in other contexts not completely disjoint from the present paper, see e.g. Harish-Chandra.

consider rational characters χ. Since R is simply connected χ takes real values by (3.2)b of [4].

COROLLARY. *The conclusion of Theorem* A *holds in Cases* (1), (2) *and* (3).

Hereafter, we assume the locally compact group G has a closed subgroup H and G/H has a finite G-invariant measure. Using Theorem A we have as in [4]:

THEOREM C. *If* ρ: $G \to \mathrm{Gl}(V)$ *is strongly admissible then each* H *invariant subspace of* V *is automatically* G*-invariant.*

In particular, by Theorem B it follows that the conclusion of Theorem C holds in Cases (1), (2) and (3) of Theorem B.

Via (2.4), (2.5) and (2.6) of [4] one has under the assumptions of Theorem C:

COROLLARY. *If* ρ *is irreducible so is* $\rho|H$, *the linear span of* $\rho(G)$ *equals that of* $\rho(H)$ *and the centralizer of* $\rho(H)$ *in* End V *equals that of* $\rho(G)$.

THEOREM D. *Let* G *be a* k*-algebraic subgroup of* $\mathrm{Gl}(V)$ *where* $k = \mathbf{R}$ *or* $\mathbf{C}$ *and suppose each* k*-rational representation,* ρ *is admissible. Then* H *is Zariski dense in* G.

PROOF. Consider the algebraic sugroup $H^{\#}$ of G. There exists a k-rational representation ρ of G on W such that $H^{\#} = \{ g \in G$ which leave stable a line l in $W\}$ [2]. In particular, l is $H^{\#}$ stable and therefore H stable. Since $\Lambda^r\rho$ is a k-rational representation if ρ is, the hypothesis implies that each ρ is strongly admissible. By Theorem C, l is G-stable. This means $H^{\#} = G$.

We see by Theorem D and Theorem B that since $\mathbf{C}$-rational representations are holomorphic that:

COROLLARY. *If* G *is minimally almost periodic or a complex analytic linear group or a real analytic linear group as in Theorem* B(3) *then* $H^{\#} = G$.

The last statement generalizes (3.4) of [4] from lattices to arbitrary cofinite volume subgroups.

In closing we note that using algebraic geometry, a different generalization of the Borel density theorem was presented by S. P. Wang in [6].

ADDED IN PROOF. It has come to the attention of the author that M. S. Raghunathan has also given a proof of the density theorem in the same simple case. Very recently another variant of the density theorem, using ergodic theory, in which a number of the present author's results (but now for S-subgroups) are reproven has been given by S. Dani.

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