ON A BOUNDEDNESS CONDITION FOR OPERATORS WITH A SINGLETON SPECTRUM

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Abstract. For a bounded invertible linear operator $A$ let $\mathcal{B}_A$ consist of those operators $X$ for which $\sup\{\|A^nXA^{-n}\|: n > 0\} > \infty$. It is shown that $\mathcal{B}_A$ contains the ideal of compact operators if and only if $A$ is similar to a scalar multiple of a unitary operator. Also, if $A$ is invertible and either has a one-point spectrum or is positive definite then $\mathcal{B}_A \cap \mathcal{B}_{A^{-1}}$ is the commutant of $A$.

In [2] Deddens shows that if $A = \int \lambda \ dE_\lambda$ is a positive invertible operator on a separable Hilbert space $\mathcal{H}$ then the nest algebra associated with the nest $\{E[0, \lambda]: \lambda > 0\}$ coincides with the set $\mathcal{B}_A$ of operators $X$ for which $\sup\{\|A^nXA^{-n}\|: n > 0\} < \infty$. Conversely every nest algebra is a $\mathcal{B}_A$ for some $A$. His results suggest that the boundedness condition defining $\mathcal{B}_A$ is of interest for any invertible $A$.

The present paper has three goals, namely to decide when $\mathcal{B}_A$ contains the compact operators, to simplify the discussion of the case in which $\mathcal{B}_A = \{A\}'$ with $\mathcal{H}$ finite dimensional, and to give a partial resolution of the same problem in the general case: $\mathcal{B}_A \cap \mathcal{B}_{A^{-1}} = \{A\}'$ if the spectrum of $A$ is a singleton.

It is a pleasure to acknowledge an indirect conversation with A. L. Shields which put me onto a theorem of Cartwright (see [1, 10.2.1]) which was a major ingredient in the original proof of Lemma 2. The simple argument given below was inspired by a comment of J. A. Deddens about a detail of that proof. I am grateful to him for several conversations about the results of this note.

We begin by resolving a question raised in [2]. (The referee mentions that the same result has been obtained by J. Stampfli and also by D. Herrero by different methods.)

Theorem 1. $\mathcal{B}_A$ contains all the compact operators for some operator $A$ if and only if $A$ is similar to a scalar multiple of a unitary operator.

Proof. Let $\alpha_n(X) = A^nXA^{-n}$ for $n > 0$. If $\sup\{\|\alpha_n(K)\|: n > 0\} < \infty$ for each compact operator $K$ then the linear transformations $\alpha_n$ are uniformly bounded on the Banach space of compact operators by the Banach-Steinhaus theorem and $\|\alpha_n\| < M$. But then if $f \otimes g$ denotes the rank one operator that takes $h \in \mathcal{H}$ to $(h, g)f$ we have $\|A^nf\|\|A^*g\| = \|A^nf \otimes A^*g\| = \|\alpha_n(f \otimes g)\| < M$ for all unit vectors $f$ and $g$. This gives $\|A^n\|\|A^{-n}\| < M$.
for \( n > 0 \). In [2] it is shown that this condition implies that \( A \) is similar to a scalar multiple of a unitary operator.

The converse assertion is clear.

In general one has \( \mathfrak{B}_A \supseteq \{ A \}' \), the commutant of \( A \). In [2] it is shown that equality holds with \( \dim \mathfrak{K} < \infty \) if and only if \( A \) is a nonzero scalar multiple of an operator of the form \( 1 + N \), \( N \) nilpotent. The necessity of this condition is also an immediate consequence of the next result.

**Lemma 1.** Let \( A \) be a bounded operator with \( \mathfrak{B}_A = \{ A \}' \). If \( \lambda \) is an eigenvalue of \( A \), \( \bar{\mu} \) an eigenvalue of \( A^* \), and if \( |\lambda| < |\mu| \) then \( \lambda = \mu \).

**Proof.** Suppose \( f \) and \( g \) are unit vectors with \( Af = \lambda f \), \( A^* \mu = \bar{\mu} g \). Then \( A^n(f \otimes g)A^{-n} = \lambda^n \mu^{-n}(f \otimes g) \) is bounded for \( n > 0 \), hence \( f \otimes g \) commutes with \( A \) and this implies \( \lambda = \mu \).

We now give a simpler proof that the condition \( A = 1 + \text{nilpotent} \) is sufficient for \( \mathfrak{B}_A = \{ A \}' \). Observe that \( A = 1 + Q' \) with \( Q' \) nilpotent (quasinilpotent) if and only if \( A = e^Q \) with \( Q \) nilpotent (quasinilpotent). Moreover,

\[
e^{\lambda Q} X e^{-\lambda Q} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \delta_n^Q(X), \quad \lambda \in \mathbb{C},
\]

where \( \delta_Q \) is the operator on \( \mathfrak{B}(\mathfrak{K}) \) given by \( \delta_Q(X) = QX - XQ \). Now if \( Q \) is nilpotent so is \( \delta_Q \), hence the entire function \( e^{\lambda Q} X e^{-\lambda Q} \) reduces to a polynomial and is therefore bounded on the positive integers only if it is constant, that is, only if \( \delta_Q(X) = 0 \) and \( X \in \{ Q \}' = \{ A \}' \).

In the remainder of this paper we are concerned with sharpening the preceding argument to see to what extent \( A = 1 + Q \), \( Q \) quasinilpotent, is sufficient for \( \mathfrak{B}_A = \{ A \}' \).

**Lemma 2.**

\[
\sup \{ \| e^{tB} X e^{-tB} \| : t > 0 \} < e^{2\|B\|} \sup \{ \| e^{nB} X e^{-nB} \| : n = 1, 2, \ldots \}
\]

for any operators \( X \) and \( B \).

**Proof.** Each positive real number \( t = n + r \) with \( 0 < r < 1 \), and \( e^{rB} \) has norm at most \( e^{\|B\|} \).

The preceding lemma gives another proof of the result of [2] that \( \mathfrak{B}_A \cap \mathfrak{B}_{A^{-1}} = \{ A \}' \) for \( A \) positive and invertible. Write \( A = e^B \) with \( B \) Hermitian. If \( X \) belongs to \( \mathfrak{B}_A \cap \mathfrak{B}_{A^{-1}} \) and \( \lambda = t + is \) then

\[
\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \delta_n^B(X) = e^{isB} e^{tB} X e^{-tB} e^{-isB}
\]

has norm at most \( \sup \{ \| e^{tB} X e^{-tB} \| : t \in \mathbb{R} \} < \infty \) so that, by Liouville's Theorem, the entire function on the left is constant and

\[
\delta_B(X) = 0, \quad X \in \{ B \}' \subseteq \{ A \}'.
\]

In [2] it is conjectured that if \( A = 1 + Q \) with \( Q \) quasinilpotent then \( \mathfrak{B}_A = \{ A \}' \). We are unable to prove this. The next result, however, shows that \( \mathfrak{B}_A \cap \mathfrak{B}_{A^{-1}} = \{ A \}' \).

**Theorem 2.** If \( Q \) is quasinilpotent and \( X \) is an operator for which \( \sup \{ \| e^{nQ} X e^{-nQ} \| : n = \pm 1, \pm 2, \ldots \} < \infty \) then \( QX = XQ \).
Proof. Let \( g_\phi(z) = \phi(e^{izQ}Xe^{-izQ}) \) for \( \phi \) a linear functional of norm 1 on \( \mathcal{B}(\mathcal{K}) \). Then \( g_\phi \) is an entire function of order

\[
\rho = \lim_{r \to \infty} \frac{\log \log M_\phi(r)}{\log r} \leq \lim_{r \to \infty} \frac{\log 4\|Q\| + \log r}{\log r} = 1.
\]

(Here \( M_\phi(r) \) is the maximum of \( |g_\phi| \) on \(|z| < r\).)

Suppose that \( g = g_\phi \) has order \( \rho < 1 \). If we choose \( \alpha \) with \( \rho < \alpha < 1 \) and \( |g(z)| \leq K \exp(r^\alpha) \) for \(|z| = r \) large enough. Let \( f(z) = g(iz) \). Then \( |f(z)| \leq C \exp(r^\alpha) \) for \( z \) in the open right half-plane \( H^+ \). Also, \( |f(iy)| \leq M < \infty \) for all real \( y \) by Lemma 2. It follows by a standard Phragmén-Lindelöf argument (see [3, p. 282]) that \( |f(z)| \leq M \) in \( H^+ \). Thus \( |g(z)| \leq M \) in the lower half-plane. The same argument applied to \( g(iz) \) shows that \( g \) is bounded on \( \mathbb{C} \).

Therefore \( 0 = g_\phi(0) = \phi(QX - XQ) \).

It remains to consider the case in which \( g_\phi \) has order 1. In this case we claim that \( g_\phi \) has type 0. To prove this, let \( \epsilon > 0 \) and choose \( N \) so that \( \|\delta^n\| < \epsilon^n \) for \( n > N \). Then

\[
|g_\phi(z)| \leq \sum_{n=0}^N \frac{r^n}{n!} |\phi(\delta^n(X))| + \sum_{n=N+1}^\infty \frac{r^n}{n!} \epsilon^n \|X\| \leq \|X\| \left( K \cdot \sum_{n=0}^N r^n + \epsilon^n \right) < \|X\|(K + 1)e^n
\]

for \(|z| = r \) large enough. It follows that the type \( \tau \) of \( g_\phi \) satisfies \( \tau = \lim_{r \to \infty} \log M_\phi(r)/r \leq \epsilon \). Hence \( \tau = 0 \) as \( \epsilon \) is arbitrary.

Therefore \( g_\phi \) is of zero exponential type and is bounded on \( \mathbb{Z} \), hence is constant (see [1, 10.2.11]).

We have \( \phi(QX - XQ) = 0 \) for all \( \phi \in \mathcal{B}(\mathcal{K})^* \) and so \( QX - XQ = 0 \).

If \( \mathcal{K} \) is finite-dimensional then \( \mathcal{B}(\mathcal{K}) = \mathcal{R}(\delta_A) \oplus (A^*)^* \) for any operator \( A \), where the indicated orthogonality is with respect to the (trace) inner product on \( \mathcal{B}(\mathcal{K}) \). By considering \( \mathcal{R}_A \) for \( A = 1 + Q \) one obtains a characterization of \( \{Q\}' \), for \( Q \) nilpotent, from the preceding results. It may be of interest to determine the corresponding characterization of the range of nilpotent and quasinilpotent derivations \( \delta_Q \).

We conclude by mentioning that, as noted in [2], Theorem 2 implies that \( \mathcal{B}_A \cap \mathcal{B}_A^{-1} = \{A\}' \) for any operator \( A \) that is quasisimilar to an operator of the form \( 1 + Q \). \( Q \) quasinilpotent. This fact would seem to discourage the search for a converse result. By certainly one can find a stronger necessary condition than that of the above lemma.

Note added. After this paper was completed I learned that Paul Roth has found a quasinilpotent \( Q \) such that \( \mathcal{B}_A \neq \{A\}' \) for \( A = 1 + Q \).

References


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