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ON A BOUNDEDNESS CONDITION FOR OPERATORS WITH A SINGLETON SPECTRUM

J. P. WILLIAMS

ABSTRACT. For a bounded invertible linear operator A let $\mathfrak{B}_A$ consist of those operators X for which $\sup\{\|A^nXA^{-n}\|: n \geqslant 0\} > \infty$. It is shown that $\mathfrak{B}_A$ contains the ideal of compact operators if and only if A is similar to a scalar multiple of a unitary operator. Also, if A is invertible and either has a one-point spectrum or is positive definite then $\mathfrak{B}_A \cap \mathfrak{B}_{A^{-1}}$ is the commutant of A.

In [2] Deddens shows that if $A = \int \lambda \, dE_\lambda$ is a positive invertible operator on a separable Hilbert space $\mathcal{H}$ then the nest algebra associated with the nest $\{E[0, \lambda]: \lambda \geqslant 0\}$ coincides with the set $\mathfrak{B}_A$ of operators X for which $\sup\{\|A^nXA^{-n}\|: n \geqslant 0\} < \infty$. Conversely every nest algebra is a $\mathfrak{B}_A$ for some A. His results suggest that the boundedness condition defining $\mathfrak{B}_A$ is of interest for any invertible A.

The present paper has three goals, namely to decide when $\mathfrak{B}_A$ contains the compact operators, to simplify the discussion of the case in which $\mathfrak{B}_A = \{A\}'$ with $\mathcal{H}$ finite dimensional, and to give a partial resolution of the same problem in the general case: $\mathfrak{B}_A \cap \mathfrak{B}_{A^{-1}} = \{A\}'$ if the spectrum of A is a singleton.

It is a pleasure to acknowledge an indirect conversation with A. L. Shields which put me onto a theorem of Cartwright (see [1, 10.2.1]) which was a major ingredient in the original proof of Lemma 2. The simple argument given below was inspired by a comment of J. A. Deddens about a detail of that proof. I am grateful to him for several conversations about the results of this note.

We begin by resolving a question raised in [2]. (The referee mentions that the same result has been obtained by J. Stampfli and also by D. Herrero by different methods.)

THEOREM 1. *$\mathfrak{B}_A$ contains all the compact operators for some operator A if and only if A is similar to a scalar multiple of a unitary operator.*

PROOF. Let $\alpha_n(X) = A^nXA^{-n}$ for $n \geqslant 0$. If $\sup\{\|\alpha_n(K)\|: n \geqslant 0\} < \infty$ for each compact operator K then the linear transformations α_n are uniformly bounded on the Banach space of compact operators by the Banach-Steinhaus theorem and $\|\alpha_n\| \leqslant M$. But then if $f \otimes g$ denotes the rank one operator that takes $h \in \mathcal{H}$ to $(h, g)f$ we have $\|A^nf\| \, \|A^{*-n}g\| = \|A^nf \otimes A^{*-n}g\| = \|\alpha_n(f \otimes g)\| \leqslant M$ for all unit vectors f and g. This gives $\|A^n\| \, \|A^{-n}\| \leqslant M$

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for $n \geqslant 0$. In [2] it is shown that this condition implies that A is similar to a scalar multiple of a unitary operator.

The converse assertion is clear.

In general one has $\mathfrak{B}_A \supseteq \{A\}'$, the commutant of A. In [2] it is shown that equality holds with dim $\mathcal{H} < \infty$ if and only if A is a nonzero scalar multiple of an operator of the form $1 + N$, N nilpotent. The necessity of this condition is also an immediate consequence of the next result.

Lemma 1. *Let A be a bounded operator with $\mathfrak{B}_A = \{A\}'$. If λ is an eigenvalue of A, $\bar{\mu}$ an eigenvalue of A^*, and if $|\lambda| \leqslant |\mu|$ then $\lambda = \mu$.*

Proof. Suppose f and g are unit vectors with $Af = \lambda f$, $A^*\mu = \bar{\mu}g$. Then $A^n(f \otimes g)A^{-n} = \lambda^n \mu^{-n}(f \otimes g)$ is bounded for $n \geqslant 0$, hence $f \otimes g$ commutes with A and this implies $\lambda = \mu$.

We now give a simpler proof that the condition $A = 1 +$ nilpotent is sufficient for $\mathfrak{B}_A = \{A\}'$. Observe that $A = 1 + Q'$ with Q' nilpotent (quasinilpotent) if and only if $A = e^Q$ with Q nilpotent (quasinilpotent). Moreover,

$$e^{\lambda Q}Xe^{-\lambda Q} = \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \delta_Q^n(X), \qquad \lambda \in \mathbf{C},$$

where δ_Q is the operator on $\mathfrak{B}(\mathcal{H})$ given by $\delta_Q(X) = QX - XQ$. Now if Q is nilpotent so is δ_Q, hence the entire function $e^{\lambda Q}Xe^{-\lambda Q}$ reduces to a polynomial and is therefore bounded on the positive integers only if it is constant, that is, only if $\delta_Q(X) = 0$ and $X \in \{Q\}' = \{A\}'$.

In the remainder of this paper we are concerned with sharpening the preceding argument to see to what extent $A = 1 + Q$, Q quasinilpotent, is sufficient for $\mathfrak{B}_A = \{A\}'$.

Lemma 2.

$$\sup\{\|e^{tB}Xe^{-tB}\|: t \geqslant 0\} \leqslant e^{2\|B\|}\sup\{\|e^{nB}Xe^{-nB}\|: n = 1, 2, \ldots\}$$

for any operators X and B.

Proof. Each positive real number $t = n + r$ with $0 \leqslant r < 1$, and e^{rB} has norm at most $e^{\|B\|}$.

The preceding lemma gives another proof of the result of [2] that $\mathfrak{B}_A \cap \mathfrak{B}_{A^{-1}} = \{A\}'$ for A positive and invertible. Write $A = e^B$ with B Hermitian. If X belongs to $\mathfrak{B}_A \cap \mathfrak{B}_{A^{-1}}$ and $\lambda = t + is$ then

$$\sum_0^{\infty} \frac{\lambda^n}{n!} \delta_B^n(X) = e^{isB}e^{tB}Xe^{-tB}e^{-isB}$$

has norm at most $\sup\{\|e^{tB}Xe^{-tB}\|: t \in \mathbf{R}\} < \infty$ so that, by Liouville's Theorem, the entire function on the left is constant and

$$\delta_B(X) = 0, \qquad X \in \{B\}' \subseteq \{A\}'.$$

In [2] it is conjectured that if $A = 1 + Q$ with Q quasinilpotent then $\mathfrak{B}_A = \{A\}'$. We are unable to prove this. The next result, however, shows that $\mathfrak{B}_A \cap \mathfrak{B}_{A^{-1}} = \{A\}'$.

Theorem 2. *If Q is quasinilpotent and X is an operator for which $\sup\{\|e^{nQ}Xe^{-nQ}\|: n = \pm 1, \pm 2, \ldots\} < \infty$ then $QX = XQ$.*

PROOF. Let $g_\phi(z) = \phi(e^{zQ}Xe^{-zQ})$ for ϕ a linear functional of norm 1 on $\mathfrak{B}(\mathcal{H})$. Then g_ϕ is an entire function of order

$$\rho = \overline{\lim_{r\to\infty}} \frac{\log\log M_\phi(r)}{\log r} \leqslant \lim_{r\to\infty} \frac{\log 4\|Q\| + \log r}{\log r} = 1.$$

(Here $M_\phi(r)$ is the maximum of $|g_\phi|$ on $|z| \leqslant r$.)

Suppose that $g = g_\phi$ has order $\rho < 1$. If we choose α with $\rho < \alpha < 1$ and $|g(z)| \leqslant K\exp(r^\alpha)$ for $|z| = r$ large enough. Let $f(z) = g(-iz)$. Then $|f(z)| \leqslant C\exp(r^\alpha)$ for z in the open right half-plane H^+. Also, $|f(iy)| \leqslant M < \infty$ for all real y by Lemma 2. It follows by a standard Phragmén-Lindelöf argument (see [3, p. 282]) that $|f(z)| \leqslant M$ in H^+. Thus $|g(z)| \leqslant M$ for z in the lower half-plane. The same argument applied to $\overline{g(\bar{z})}$ shows that g is bounded on $\mathbf{C}$. Therefore $0 = g_\phi'(0) = \phi(QX - XQ)$.

It remains to consider the case in which g_ϕ has order 1. In this case we claim that g_ϕ has type 0. To prove this, let $\varepsilon > 0$ and choose N so that $\|\delta_Q^n\| < \varepsilon^n$ for $n \geqslant N$. Then

$$|g_\phi(z)| \leqslant \sum_{n=0}^{N} \frac{r^n}{n!}|\phi(\delta_Q^n(X))| + \sum_{n=N+1}^{\infty} \frac{r^n}{n!}\varepsilon^n\|X\|$$
$$\leqslant \|X\|\left(K\cdot\sum_0^N r^n + e^{\varepsilon r}\right) \leqslant \|X\|(K+1)e^{\varepsilon r}$$

for $|z| = r$ large enough. It follows that the type τ of g_ϕ satisfies $\tau = \overline{\lim}_r[\log M_\phi(r)/r] \leqslant \varepsilon$. Hence $\tau = 0$ as ε is arbitrary.

Therefore g_ϕ is of zero exponential type and is bounded on $\mathbf{Z}$, hence is constant (see [1, 10.2.11]).

We have $\phi(QX - XQ) = 0$ for all $\phi \in \mathfrak{B}(\mathcal{H})^*$ and so $QX - XQ = 0$.

If $\mathcal{H}$ is finite-dimensional then $\mathfrak{B}(\mathcal{H}) = \mathfrak{R}(\delta_A) \oplus \{A^*\}'$ for any operator A, where the indicated orthogonality is with respect to the (trace) inner product on $\mathfrak{B}(\mathcal{H})$. By considering $\mathfrak{B}_A$ for $A = 1 + Q$ one obtains a characterization of $\{Q\}'$, for Q nilpotent, from the preceding results. It may be of interest to determine the corresponding characterization of the range of nilpotent and quasinilpotent derivations δ_Q.

We conclude by mentioning that, as noted in [2], Theorem 2 implies that $\mathfrak{B}_A \cap \mathfrak{B}_{A^{-1}} = \{A\}'$ for any operator A that is quasisimilar to an operator of the form $1 + Q$, Q quasinilpotent. This fact would seem to discourage the search for a converse result. By certainly one can find a stronger necessary condition than that of the above lemma.

NOTE ADDED. After this paper was completed I learned that Paul Roth has found a quasinilpotent Q such that $\mathfrak{B}_A \neq \{A\}'$ for $A = 1 + Q$.

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DEPARTMENT OF MATHEMATICS, INDIANA UNIVERSITY, BLOOMINGTON, INDIANA 47401