THE WEAK BEHAVIOR OF SPHERICAL MEANS

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ABSTRACT. C. Fefferman has shown that the disc multiplier is not bounded on $L^p(\mathbf{R}^n)$, $n > 1$, $p \neq 2$. In contrast, C. Herz showed that, when restricted to L^p radial functions, it is bounded on $L^p(\mathbf{R}^n)$ if and only if $2n/(n + 1) < p < 2n/(n - 1)$. We show that it is not weakly bounded for $p = 2n/(n + 1)$ or $p = 2n/(n - 1)$.

Define the linear operator T on $L^2(\mathbf{R}^n)$ by

$$\widehat{Tf}(y) = \begin{cases} \hat{f}(y) & \text{if } |y| < 1, \\ 0 & \text{if } |y| > 1. \end{cases}$$

We shall assume throughout the remainder of this paper that $n > 1$. Charles Fefferman [1] has shown that T is not bounded on $L^p(\mathbf{R}^n)$ for any $p \neq 2$. In contrast, Herz [2] showed that if one is concerned only with L^p radial functions, T is bounded on $L^p(\mathbf{R}^n)$ if and only if $2n/(n + 1) < p < 2n/(n - 1)$. We shall establish the following result.

THEOREM. *T is not weakly bounded on $L^p(\mathbf{R}^n)$ radial functions for $p = 2n/(n + 1)$.*

The proof is quite simple. Following Herz, note first that if f is a radial function on $\mathbf{R}^n$, $\hat{f}$ is also radial, and

$$\hat{f}(R) = 2\pi \int_0^\infty \frac{J_{(n-2)/2}(2\pi rR)}{|rR|^{(n-2)/2}} f(r) r^{n-1} \, dr.$$

Therefore

$$Tf(s) = \frac{4\pi^2}{s^{(n-2)/2}} \int_0^\infty r^{n/2} f(r) \int_0^1 J_{(n-2)/2}(2\pi sR) J_{(n-2)/2}(2\pi rR) R \, dR \, dr.$$

Denote the inner integral by $K(s, r)$. From Watson [3, p. 46], we see

$$\left(x^k J_k(x)\right)' = x^k J_{k-1}(x), \qquad \left(x^{-k} J_k(x)\right)' = -x^{-k} J_{k+1}(x).$$

Employing these in an integration by parts in the expression for K, we see

$$2\pi K(s, r) = \frac{1}{r} J_{(n-2)/2}(2\pi s) J_{n/2}(2\pi r) + 2\pi \frac{s}{r} \int_0^1 J_{n/2}(2\pi sR) J_{n/2}(2\pi rR) R \, dR.$$

Noting the symmetry of K in r and s, we compute $2\pi K(s, r)(r/s - s/r)$, and

obtain from this the result

$$Tf(s) = \frac{2\pi}{s^{(n-2)/2}} \int_0^\infty s \frac{J_{(n-2)/2}(2\pi r)J_{n/2}(2\pi s)}{s^2 - r^2} f(r)r^{n/2} \, dr$$

$$- \frac{2\pi}{s^{(n-2)/2}} \int_0^\infty r \frac{J_{(n-2)/2}(2\pi s)J_{n/2}(2\pi r)}{s^2 - r^2} f(r)r^{n/2} \, dr$$

$$= T_0 f(s) + T_1 f(s).$$

We shall let

$$f(r) = \frac{J_{(n-2)/2}(2\pi r)}{r^{n/2}} \chi_{(1,\beta)}(r)$$

for large β, and shall evaluate $Tf(s)$ for $s > 2\beta$. On such functions, T_1 is an operator which is weakly bounded:

$$|T_1 f(s)| \leqslant c \frac{|J_{(n-2)/2}(2\pi s)|}{s^{(n-2)/2}} \int_0^{s/2} \frac{|J_{n/2}(2\pi r)|}{s^2 - r^2} rf(r)r^{n/2} \, dr.$$

Using the asymptotic relationships

$$J_k(r) = \left(\frac{2}{\pi r}\right)^{1/2} \cos\left(r - \frac{\pi k}{2} - \frac{\pi}{4}\right) + O\left(\frac{1}{r^{3/2}}\right), \qquad r \to \infty,$$

$$\sim cr^k, \qquad r \to 0,$$

we see that

$$|T_1 f(s)| \leqslant \frac{c}{s^{(n+1)/2}s} \left(\int_0^{s/2} \left|\frac{J_{n/2}(2\pi r)}{r^{n/2}} r^2\right|^{2n/(n-1)} r^{n-1} \, dr\right)^{(n-1)/2n} \|f\|_{2n/(n+1)}$$

$$\leqslant \frac{c\|f\|_{2n/(n+1)}}{s^{(n+1)/2}}.$$

Thus T_1 is weakly bounded at $2n/(n+1)$, with respect to the measure $d\mu = r^{n-1}dr$, and to prove the result it suffices to show T_0 is not. But for the given f, the asymptotics for the Bessel functions show that for $s > 2\beta$,

$$|T_0 f(s)| = c_0 \frac{|J_{n/2}(2\pi s)|}{s^{(n-2)/2}} \int_1^\beta |J_{(n-2)/2}(2\pi r)|^2 \frac{s}{s^2 - r^2} \, dr$$

$$\geqslant c_1 \frac{|J_{n/2}(2\pi s)|}{s^{n/2}} \int_1^\beta \frac{|\cos(2\pi r - \pi(n-3)/4)|^2}{r} \, dr$$

$$\geqslant c_2 \frac{|J_{n/2}(2\pi s)|}{s^{n/2}} \log \beta.$$

Therefore for β large,

$$\mu\{s \geqslant 2\beta \mid |T_0 f(s)| > \alpha\} \geqslant c_3 \mu\left\{s > 2\beta \mid c_2 \frac{\log \beta}{\alpha} > s^{(n+1)/2}\right\}.$$

Choose $\alpha = (\log \beta)/c_4 \beta^{(n+1)/2}$; then

$$\mu\{s \mid |T_0 f(s)| > \alpha\} \geqslant c_3 \int_{2\beta}^{c_5 \beta} s^{n-1} \, ds,$$

where $c_5 = (c_2 c_4)^{2/(n+1)}$ is chosen so that $(c_5^n - 2^n) c_3 / n = 1$. Thus, all in all,

$$\mu\{s| \ |T_0 f(s)| > \alpha\} \geqslant \beta^n.$$

But if T_0 is weakly bounded,

$$\beta^n \leqslant \mu\{s| \ |T_0 f(s)| > \alpha\} \leqslant c_6 \left(\frac{\|f\|_{2n/(n+1)}}{\alpha} \right)^{2n/(n+1)}$$

$$\leqslant c_7 \log \beta \left(\frac{\log \beta}{c_4 \beta^{(n+1)/2}} \right)^{-2n/(n+1)} = c_8 \beta^n (\log \beta)^{-(n-1)/(n+1)}.$$

This is a contradiction.

The same counterexample can be used to show that T is not weakly bounded on $L^p(\mathbf{R}^n)$ if $p = 2n/(n-1)$, by showing the adjoint of T is not bounded from $L^{p',1}$ to $L^{p'}$.

References

1. C. Fefferman, *The multiplier problem for the ball*, Ann. of Math. **94** (1971), 330–336.
2. C. Herz, *On the mean inversion of Fourier and Hankel transforms*, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 996–999.
3. G. N. Watson, *Theory of Bessel functions*, Cambridge Univ. Press, London, 1966.

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