THERE ARE NO $Q$-POINTS IN LAVER’S MODEL
FOR THE BOREL CONJECTURE

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ABSTRACT. It is shown that it is consistent with ZFC that no nonprincipal
ultrafilter on $\omega$ is a $Q$-point (also called a rare ultrafilter).

All ultrafilters are assumed to be nonprincipal and on $\omega$.

Definitions. (1) $U$ is a $Q$-point (also called rare [C]) iff $\forall f \in \omega^\omega$ if $f$ is
finite-to-one then $\exists X \in U, f \upharpoonright X$ is one-to-one.
(2) $U$ is a $P$-point iff $\forall f \in \omega^\omega, \exists X \in U, f \upharpoonright X$ is constant or finite-to-one.
(3) $U$ is a semi-$Q$-point (also called rapid [C], if $\forall f \in \omega^\omega, \exists g \in \omega^\omega, \forall n f(n) < g(n)$ and $g''\omega \in U$.
(4) $U$ is semiselective iff it is a $P$-point and a semi-$Q$-point.
(5) For $f, g \in \omega^\omega, [f < g$ iff $\exists n \forall m > n (f(m) < g(m))]$.
(6) For $S \subseteq \omega^\omega, [S$ is dominant iff $\forall f \in \omega^\omega \exists g \in S (f < g)]$.

Theorem 1 (Ketonen [Ke]). If every dominant family has cardinality $2^{\aleph_0}$,
then there exists a $P$-point.

Theorem 2 (Mathias, Taylor [M3]). If there exists a dominant family of
cardinality $\aleph_1$, then there exists a $Q$-point.

Kunen [Ku1] showed that adding $\aleph_2$ random reals to a model of ZFC +
GCH gives a model with no semiselective ultrafilters. More recently he
showed [Ku2] that if one first adds $\aleph_1$ Cohen reals (then the random reals)
then the resulting model has a $P$-point. In either case one has a dominant
family of size $\aleph_1$ so there is a $Q$-point.

Theorem 3. The following are equivalent:
(1) $U$ is a semi-$Q$-point.
(2) Given $P_n \subseteq \omega$ finite for $n < \omega$ there exists $X \in U$ such that $\forall n, |X \cap P_n| < n$.
(3) $\exists h \in \omega^\omega$ such that given $P_n \subseteq \omega$ finite for $n < \omega$ there exists $X \in U$ such
that $\forall n, |X \cap P_n| < h(n)$.
Proof. (1) $\Rightarrow$ (2). Let $f(n) = \sup(\bigcup_{m \leq n} P_m) + 1$. Suppose that for all $n$, $g(n) > f(n)$; then $P_n \cap g''\omega \subseteq \{ g(0), \ldots, g(n - 1) \}$.
(3) $\Rightarrow$ (1). Assume $f$ increasing. Choose $n_0 < n_1 < n_2 < \cdots$, so that $h(k + 1) < n_k$. Let $P_k = f(n_k)$ and let $Y \in U$ so that $|Y \cap P_k| < h(k)$. Then, for each $m > n_0, |Y \cap f(m)| < m$, since if $n_k \leq m < n_{k+1}$ then

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Hence if \( g \in \omega^\omega \) enumerates \( Y - f(n_0 + 1) \) in increasing order then \( \forall n, f(n) < g(n) \). □

Define \( U \times V = \{ A \subseteq \omega \times \omega: \{ n: \{ m: (n, m) \in A \} \in V \} \in U \} \). Whilst \( U \times V \) is never a \( P \)-point or a \( Q \)-point, nevertheless:

**Theorem 4.** \( U \times V \) is a semi-\( Q \)-point iff \( V \) is a semi-\( Q \)-point.

**Proof.** (\( \Rightarrow \)) Given \( P_k \subseteq \omega \) finite let \( P_k^* = \{ \langle n, m \rangle: m \in P_k \) and \( n < m \)\). Choose \( Z \in U \times V \) so that \( \forall k, |Z \cap P_k^*| < k \). Let \( n \in \omega \) so that \( Y = \{ m > n: (n, m) \in Z \} \subseteq V \) then \( \forall k, |Y \cap P_k| < k \). (More generally if \( f: U \rightarrow V \) and \( U \) is a semi-\( Q \)-point and \( f \) is finite-to-one then \( V \) is a semi-\( Q \)-point.)

(\( \Leftarrow \)) Given \( P_k \subseteq \omega^2 \) finite, choose \( n_k \) increasing so that \( P_k \subseteq n_k \). Let \( Y \in V \) so that \( \forall k, |n_k \cap Y| < k \). Let \( Z = \bigcup_{k<\omega}(k) \times \{ m: m \in Y \) and \( m > n_k \} \)

which has cardinality \( < (k + 1)^2 \). □

**Theorem 5.** In Laver’s model \( N \) for the Borel conjecture \([L]\) there are no semi-\( Q \)-points.

**Proof.** Some definitions from \([L]\):

1. \( T \in \mathcal{F} \) iff \( T \) is a subtree of \( \omega^{<\omega} \) with the property that there exist \( s \in T \) (called stem \( T \)) so that \( \forall t \in T, t \subseteq s \) or \( s \subseteq t \), and if \( t \supseteq s \) and \( t \in T \) then there are infinitely many \( n \in \omega \) such that \( t\langle n \rangle \in T \).

2. \( T^* \succ T \) iff \( \hat{T} \subseteq T \).

3. \( T_s = \{ t \in T: s \subseteq t \) or \( t \subseteq s \} \).

4. \( T^0 \succ \hat{T} \) iff \( T \succ \hat{T} \) and they have the same stem.

5. For \( x < y < \omega \) let \( [x, y) = \{ n < \omega: x < n < y \} \).

**Lemma 1.** Suppose we are given \( T \in \mathcal{F} \) and finite sets \( F_s \) for each \( s \in T - \emptyset \) such that for each \( s \in T - \emptyset \):

- (a) if \( s = (k_0, \ldots, k_m, k_{n+1}) \), then \( F_s \subseteq [k_m, k_{n+1}) \);
- (b) if \( s = \langle n \rangle \), then \( F_s \subseteq [0, n) \);
- (c) \( \exists N < \omega \forall t \) immediately below \( s \) in \( T|F_s| < N \). For any \( \hat{T} \succ T \) let \( H_s = \bigcup \{ F_s: s \in \hat{T} \} \). Then \( \exists T^1, T^0 \supseteq T \) such that \( H_{T^0} \cap H_{T^1} \) is finite.

**Proof.** We may as well assume that the stem of \( T \) is \( \emptyset \). Given \( Q \) any infinite family of sets of cardinality \( N < \omega \) there exists \( G, |G| < N \), \( \exists \tilde{Q} \subseteq Q \) infinite so that \( \forall F, \tilde{F} \in \tilde{Q}, F \cap \tilde{F} \subseteq G \) (i.e., a \( \Delta \)-system). Now trim \( T \) to obtain \( \hat{T} \succ T \) so that \( \forall s \in T, \exists G_s \subseteq [k_n, \omega) \) finite \( (s = (k_0, \ldots, k_n)) \) and for all \( \hat{t} \) immediately below \( s \) in \( \hat{T} \), \( (F_s \cap F_{\hat{t}}) \subseteq G_s \). Build two sequences of finite subtrees of \( \hat{T} \):

\[
T^0_n \subseteq T^1_{n+1} \cdots, \quad T^1_n \subseteq T^0_{n+1} \cdots
\]
so that
\[ \bigcup_{s \in T_0^i} (F_s \cup G_s) \cap \bigcup_{s \in T_1^i} (F_s \cup G_s) \subseteq G_\emptyset \]
and \( \bigcup_{n < \omega} T_i^n = T_i > \hat{T} \) for \( i = 0, 1 \).

This is done as follows: Suppose we have \( T_0^0, T_1^0 \) and we are presented with \( s \in T_0^0 \) and asked to add an immediate extension of \( s \) to \( T_0^0 \). Then since \( \{ F_t - G_t: t \text{ immediately below } s \text{ in } \hat{T} \} \) is a family of disjoint sets and \( G_t \subseteq [k_n, \omega) \) where \( t = (k_0, \ldots, k_n) \) we can find infinitely many \( t \) immediately below \( s \) in \( \hat{T} \) so that
\[ [(F_t - G_t) \cup G_t] \cap \bigcup_{s \in T_*^i} (F_s \cup G_s) = \emptyset. \qquad \Box \]

The above is a double fusion argument.

Some more definitions from [L]:

(1) Fix a natural \( \omega \)-ordering of \( \omega^{<\omega} \) and for any \( T \in \mathcal{F} \) transfer it to \( \{ t \in T: \text{stem } T \subseteq t \} \) in a canonical fashion. \( T(n) \) denotes the \( n \)th element of \( \{ t \in T: \text{stem } T \subseteq t \} \) for \( t \in T \). (2) For \( T \) and \( V \), \( T < V \) and \( V < n, T(i) = V(i) \).

(3) The p.o. \( \mathbf{P}_{\omega^2} \) is the \( \omega_2 \)-iteration of \( \mathcal{F} \) with countable support (\( p \vdash \alpha \Rightarrow \forall \gamma \in \alpha \vdash \gamma(n) = f(n) \)) and \( \text{supp}(\alpha) = \{ \alpha: \forall n < \omega, \gamma(n) = f(n) \} \) is countable.

(4) For \( K \) finite and \( n < \omega \), \( p, q \in K \) such that \( p \vdash \forall i \in K, p(i) \vdash \forall n \vdash \alpha \vdash \gamma(n) \Rightarrow \gamma(n) = f(n) \), then \( p \vdash \forall i \in K, \gamma(i) = f(i) \).

**Lemma 2.** Let \( f \) be a term denoting the first Laver real and \( \tau \) any term. If \( p \in \mathbf{P}_{\omega^2} \) and \( p \vdash \tau \in \omega^n \vdash \forall n \vdash \gamma(n) < \tau(n) \) and \( \tau \) increasing then \( \exists Z_0, Z_1 \) such that \( Z_0 \cap Z_1 \) is finite.\n
**Proof.** Construct a sequence \( p < q \) so that \( \bigcup_{n < \omega} K_n = \bigcup_{n < \omega} \text{supp}(p) \) and \( 0 \in K_0 \). Having gotten \( p_n \), let \( s = (k_0, \ldots, k_m) \) be \( p_n(0)(n) \). Then for each \( i \leq m + 1 \),
\[ p_t = \langle p_n(0), \bigcap_{[1, \omega_2]} \rangle \vdash \forall t \in k_n, \tau(i) = f(i). \]

Hence by applying Lemma 6 of [L] \( m + 2 \) many times we can find \( q \in K_n \) such that \( \bigcup_{n < \omega} K_n = \bigcup_{n < \omega} \text{supp}(p) \) and \( 0 \in K_0 \). Having gotten \( p_n \), let \( s = (k_0, \ldots, k_m) \) be \( p_n(0)(n) \). Then for each \( i \leq m + 1 \),
\[ p_t = \langle p_n(0), \bigcap_{[1, \omega_2]} \rangle \vdash \forall t \in k_n, \tau(i) = f(i). \]

Therefore, \( p_{n+1} \vdash \tau \in [k_n, k_{n+1}] \subseteq F_t \). Hence by applying Lemma 6 of [L] \( m + 2 \) many times we can find \( q \in K_n \) such that \( \bigcup_{n < \omega} K_n = \bigcup_{n < \omega} \text{supp}(p) \) and \( 0 \in K_0 \). Having gotten \( p_n \), let \( s = (k_0, \ldots, k_m) \) be \( p_n(0)(n) \). Then for each \( i \leq m + 1 \),
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\[ p_t = \langle p_n(0), \bigcap_{[1, \omega_2]} \rangle \vdash \forall t \in k_n, \tau(i) = f(i). \]
Proof of Theorem 5. Suppose $M[G_{\omega_2}] \models \text{"}U \text{ is a semi-Q-point".}$. Applying an argument of Kunen's we get $\alpha < \omega_2$ such that $U \cap M[G_\alpha] \in M[G_\alpha]$. $(M[G_\beta] \models \text{"}CH\text{" for all } \beta < \omega_2$ so construct using $\omega_2$-c.c., $\alpha_\lambda < \omega_2$ for $\lambda < \omega_1$ so that $\forall x \in M[G_{\alpha_\lambda}] \cap 2^\omega$, $P_{\alpha_{\alpha_\lambda}}$ decides $\text{"}x \in U\text{"}$. Let $\alpha = \text{sup } \alpha_\lambda$. Note $M[G_\alpha] \cap 2^\omega = \bigcup_{\beta < \alpha} M[G_\beta] \cap 2^\omega$ since $\kappa_1$ is not collapsed.) By [L, Lemma 11] we may assume $U \cap M \in M$. But Lemma 2 clearly implies that for any $V \text{ ult. in } M$, $M[G_{\omega_2}] \models \text{"}no extension of } V \text{ is a semi-Q-point.}\text{"} \square$

Remarks. (1) A similar argument shows that in the model gotten by $\omega_2$ iteration of Mathias forcing with countable support there are no semi-Q-points. In fact, as Mathias later pointed out to me, the appropriate argument needed is an easy generalization of Theorem 6.9 of [M2].

(2) In [M1] Mathias shows $[\omega \rightarrow (\omega)^\omega] \Rightarrow \text{[There are no rare filters or nonprincipal ultrafilters.]}$

(3) In neither the Laver or Mathias models are there small dominant families so by Ketenen [Ke] there is a P-point. Also it is easily shown no ultrafilter is generated by fewer then $\kappa_2$ sets.

(4) Not long after the results of this paper were obtained, Shelah showed that it is consistent that no P-points exist [W]. In his model there is a dominant family of size $\kappa_1$, so there are Q-points. It remains open whether or not it is consistent that there are no P-points or Q-points.

Conjecture. Borel conjecture $\iff$ there does not exist a semi-Q-point.

References


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