GAUSSIAN MEASURE OF LARGE BALLS
IN A HILBERT SPACE

CHII-RUEY HWANG

Abstract. Let $P$ be a zero mean Gaussian measure in a Hilbert space. The asymptotic behavior of $P\{\|x - b\|^2 > \epsilon\}$ as $\epsilon \to \infty$ is studied in this note.

Let $P$ be a Gaussian measure in a separable Hilbert space $\mathcal{H}$ and $b$ be an arbitrary fixed element in $\mathcal{H}$. We shall study how fast $P\{\|x - b\|^2 > \epsilon\} \to 0$ as $\epsilon \to \infty$.

Without loss of generality, $P$ is assumed to be of mean 0. Let $B$ be the covariance operator of $P$, and let the eigenvalues (corresponding eigenvectors) of $B$ be ordered by $\lambda_1 \geq \lambda_2 \geq \cdots (\{e_i\})$. Let $k$ be the multiplicity of the largest eigenvalue, $x_i = \langle x, e_i \rangle$, $b_i = \langle b, e_i \rangle$ and $a = (\sum_1^k b_i^2)^{1/2}$. Finally, let $F$ denote the distribution function of $\|x - b\|^2 = \Sigma_1^\infty (x_i - b_i)^2$. The Laplace transform of $F$ is

$$
\varphi(c) = \int_0^\infty e^{-ct} dF(t) = \prod_{i=1}^\infty [(1 + 2c\lambda_i)^{-1/2} \exp(-b_i^2c(1 + 2c\lambda_i)^{-1})],
$$

$$
\psi(c) = \int_0^\infty e^{-ct + t/2\lambda_i}(1 - F(t)) dt
$$

$$
= \left(1 - \varphi(c - \frac{1}{2\lambda_1})\right)\left(c - \frac{1}{2\lambda_1}\right)^{-1}
$$

$$
= \left(c - \frac{1}{2\lambda_1}\right)^{-1} (2c\lambda_1)^{-k/2} \exp\left(\frac{a^2}{4c\lambda_1^2} - \frac{a^2}{2\lambda_1}\right)
$$

$$
\times \left[ (2c\lambda_1)^{k/2} \exp\left(\frac{a^2}{2\lambda_1} - \frac{a^2}{4c\lambda_1^2}\right) - \prod_{k+1}^\infty \left(1 + 2c\lambda_i - \frac{\lambda_i}{\lambda_1}\right)^{-1/2}
$$

$$
\times \exp - \sum_{k+1}^\infty b_i^2 \left(c - \frac{1}{2\lambda_1}\right)\left(1 + 2c\lambda_i - \frac{\lambda_i}{\lambda_1}\right)^{-1}\right].
$$

If $a = 0$, then

$$
\psi(c) \sim (2\lambda_1)^{1-k/2} \prod_{k+1}^\infty \left(1 - \frac{\lambda_i}{\lambda_1}\right)^{-1/2} \exp\left(\frac{1}{2\lambda_1} \sum_{k+1}^\infty b_i^2 \left(1 - \frac{\lambda_i}{\lambda_1}\right)\right)^{-1} c^{-k/2}
$$

Received by the editors February 5, 1979.

AMS (MOS) subject classifications (1970). Primary 60B99; Secondary 60G15.

Key words and phrases. Asymptotic behavior, Gaussian measure, Hilbert space, Laplace transform, Tauberian theorem.

This research was supported by NSF Grant MCS 76-80762.

© 1980 American Mathematical Society

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
as $c \to 0$. By a Tauberian theorem \cite{1}, as $\varepsilon \to \infty$,

$$
\int_0^\varepsilon e^{\varepsilon^{1/2} \lambda_1} (1 - F(t)) \, dt \sim \frac{1}{\Gamma(k/2 + 1)} (2\lambda_1)^{1-k/2} \prod_{k+1}^\infty (1 - \frac{\lambda_i}{\lambda_1})^{-1/2} \times \exp \left( \frac{1}{2\lambda_1} \sum_{k+1}^\infty b_i^2 \left( 1 - \frac{\lambda_i}{\lambda_1} \right)^{-1} \right) \varepsilon^{k/2}.
$$

Since $e^{\varepsilon^{1/2} \lambda_1} (1 - F(t))$ is continuous and positive, L'Hôpital's rule is applicable. Hence we have

**Theorem 1.** If $a = (\Sigma_i b_i^2)^{1/2} = 0$, then

$$P \{ \|x - b\|^2 > \varepsilon \} \sim K_1 e^{-\varepsilon^{1/2} \lambda_i^{k/2}}.$$

where

$$K_1 = \frac{1}{\Gamma(k/2)} (2\lambda_1)^{1-k/2} \prod_{k+1}^\infty (1 - \frac{\lambda_i}{\lambda_1})^{-1/2} \exp \left( \frac{1}{2\lambda_1} \sum_{k+1}^\infty b_i^2 \left( 1 - \frac{\lambda_i}{\lambda_1} \right)^{-1} \right).$$

(2)

If $a > 0$, then $\psi(c) \to \infty$ exponentially fast as $c \to 0$. The ordinary Tauberian theorem is not applicable here. We shall try another approach.

Let $\vec{b} \in \mathbb{R}$ such that $\vec{b}_i = 0$ if $i < k$ and $\vec{b}_i = b_i$ otherwise. Then, for $\varepsilon$ large enough

$$P \{ \|x - \vec{b}\|^2 > \varepsilon \} \leq P \{ \|x - \vec{b}\|^2 > (\sqrt{\varepsilon} - a)^2 \} \sim K_1 \left( \exp -\frac{1}{2\lambda_1} (\sqrt{\varepsilon} - a)^2 \right) e^{\frac{k}{2}-1}.$$

On the other hand, $P \{ \|x - b\|^2 > \varepsilon \}$ may be regarded as $P_b \{ \|x\|^2 > \varepsilon \}$, where $P_b$ has mean $-b$ and covariance $B$. It is easily seen that $P_b$ is equivalent to $P_\vec{b}$ and

$$\frac{dP_b}{dP_\vec{b}}(x) = \exp \left( \frac{-\Sigma_i x_i b_i}{\lambda_1} - \frac{a^2}{2\lambda_1} \right).$$

Therefore

$$P \{ \|x - b\|^2 > \varepsilon \} = P_b \{ \|x\|^2 > \varepsilon \}
= \int_{\|x\|^2 > \varepsilon} \exp \left( \frac{-\Sigma_i x_i b_i}{\lambda_1} - \frac{a^2}{2\lambda_1} \right) \, dP_\vec{b}(x)
> (2\pi \lambda_1)^{-k/2} \int_{x_1 + \ldots + x_k > \varepsilon} \exp \left( \frac{-1}{2\lambda_1} \sum_{i=1}^k (x_i + b_i)^2 \right) \, dx_1 \ldots dx_k
= (2\pi \lambda_1)^{-k/2} \int_{x_1 + \ldots + x_k > \varepsilon} \times \exp \left( \frac{-1}{2\lambda_1} \left[ (x_1 - a)^2 + x_2^2 + \ldots + x_k^2 \right] \right) \, dx_1 \ldots dx_k.$$

(3)
GAUSSIAN MEASURE OF LARGE BALLS 109

(To get the last equality, just rotate the vector \((-b_1, \ldots, -b_k)\) to the first coordinate.)

For \(k > 3\), change the last \(k - 1\) coordinates to polar coordinates; (3) becomes

\[
(2\pi\lambda_1)^{-k/2} \frac{2\pi^{(k-1)/2}}{\Gamma((k-1)/2)} \int_{r > 0} \frac{\exp - \frac{(x_1 - a)^2 + r^2}{2\lambda_1}}{r^{k-2}} dr \, dx_1
\]

\[
= (2\pi\lambda_1)^{-k/2} \frac{2\pi^{(k-1)/2}}{\Gamma((k-1)/2)}
\times \int_0^{\pi} \int_0^{\infty} \left( \exp \frac{\rho^2 + a^2 - 2\rho a \cos \theta}{2\lambda_1} \right) \rho (\rho \sin \theta)^{k-2} d\rho \, d\theta.
\]

By L'Hôpital's rule

\[
\lim_{\epsilon \to \infty} \int_0^{\pi} \int_0^{\infty} \left( \exp \frac{2\rho a \cos \theta - \rho^2 - a^2}{2\lambda_1} \right) \rho (\rho \sin \theta)^{k-2} d\rho \, d\theta
\]

\[
= \lim_{\epsilon \to \infty} \frac{\epsilon^{(k-3)/4} \exp(- (\sqrt{\epsilon} - a)^2 / 2\lambda_1)}{\lambda_1^{-1} \epsilon^{(k-3)/4} \exp(- (\sqrt{\epsilon} - a)^2 / 2\lambda_1)}
\]

\[
= \frac{1}{2} \frac{\lambda_1}{(2\pi)^{1/2}} a^{(1-k)/2} \exp \frac{\lambda_1}{2\pi} a^{(1-k)/2}.
\]

Hence (4) is

\[
\int_0^{\pi} \exp \frac{\sqrt{\epsilon} a \cos \theta}{\lambda_1} (\sin \theta)^{k-2} d\theta \sim \frac{1}{2} \Gamma \left( \frac{k-1}{2} \right) \left( \frac{2\lambda_1}{a\sqrt{\epsilon}} \right)^{(k-1)/2} \exp \frac{\sqrt{\epsilon} a}{\lambda_1}.
\]

And, by Laplace's method [2]

\[
\int_0^{\pi} \exp \frac{\sqrt{\epsilon} a \cos \theta}{\lambda_1} (\sin \theta)^{k-2} d\theta \sim \frac{1}{2} \Gamma \left( \frac{k-1}{2} \right) \left( \frac{2\lambda_1}{a\sqrt{\epsilon}} \right)^{(k-1)/2} \exp \frac{\sqrt{\epsilon} a}{\lambda_1}.
\]

Hence (4) is

\[
\frac{1}{2} \lambda_1 \Gamma \left( \frac{k-1}{2} \right) \left( \frac{2\lambda_1}{a} \right)^{(k-1)/2},
\]

and, consequently, (3) is asymptotic to

\[
\sqrt{\frac{\lambda_1}{2\pi}} a^{(1-k)/2} \epsilon^{(k-3)/4} \exp - \frac{1}{2\lambda_1} (\sqrt{\epsilon} - a)^2.
\]

(The special cases \(k = 1\) and \(k = 2\) easily lead to the same expression.)

To sum up, we have

**Theorem 2.** If \(a > 0\), then

\[
\lim_{\epsilon \to \infty} \sup P \{ \|x - b\|^2 > \epsilon \} \epsilon^{1-k/2} \exp - \frac{1}{2\lambda_1} (\sqrt{\epsilon} - a)^2 < K_1,
\]

where \(K_1\) is defined by (2);

\[
\lim_{\epsilon \to \infty} \inf P \{ \|x - b\|^2 > \epsilon \} \epsilon^{(3-k)/4} \exp - \frac{1}{2\lambda_1} (\sqrt{\epsilon} - a)^2 > \sqrt{\frac{\lambda_1}{2\pi}} a^{(1-k)/2}.
\]
Remarks. 1. Zolotarev discussed the limiting behavior of $P\left\{ \sum_{i=1}^{\infty} x_i^2 > \epsilon \right\}$ as $\epsilon \to \infty$ in [4], where $\{x_i\}$ are independent $\mathcal{N}(0, \sigma_i^2)$ with $\sum_{i=1}^{\infty} \sigma_i^2 < \infty$. This is a special case of Theorem 1. The proof here is much simpler.

2. It is tempting to try to apply Theorem 3 of [4], but this would require establishing that $\frac{e^{\epsilon / (2 \lambda)}}{(2 \lambda)(1 - F(\epsilon))}$ is nondecreasing, and would still give a weakened version of Theorem 2. Of course, our theorem also leaves open the question of the exact asymptotic behavior of $1 - F(\epsilon)$, when $a > 0$.

Acknowledgement. The Tauberian argument was suggested by Professor Ulf Grenander.

Bibliography


Division of Applied Mathematics, Brown University, Providence, Rhode Island 02912

Current address: Institute of Mathematics, Academia Sinica, Taipai, Taiwan, Republic of China