COMPACTIFICATIONS WITH COUNTABLE REMAINDER

M. G. CHARALAMBOUS

Abstract. In this paper, we deal with the problem of characterizing those spaces that have a compactification with countable remainder.

1. Introduction and definitions. A collection $\mathcal{A}$ of subsets of a topological space $X$ is called a network if every open subset of $X$ is the union of a subcollection of $\mathcal{A}$. $R(X)$ denotes the set of all points of $X$ which possess no compact neighbourhood. If $Y$ is a Hausdorff compactification of $X$, it is readily seen that $R(X)$ is the intersection of $X$ with the closure of $Y - X$ in $Y$. A Hausdorff compactification $Y$ of $X$ is said to have countable remainder if $Y - X$ is a countable set; by an abuse of terminology, we shall say that such a $Y$ is a countable compactification of $X$. In what follows, the space $X$ is assumed to be at least Tychonoff. Two necessary conditions for $X$ to have a countable compactification are (a) $X$ is Čech-complete and (b) $X$ is rim-compact. These are, in fact, sufficient conditions as well in the case when $X$ is metric separable [6], [10]. However, the product of the space of irrational numbers with an uncountable discrete space, despite satisfying both (a) and (b), possesses no countable compactification [4]. There has recently been interest in finding conditions which, together with (a) and (b), ensure that $X$ has a countable compactification ([2], [3], [4], [8]). Terada has shown that one such condition is that $R(X)$ is compact metric, and Hoshina has weakened this to the requirement that $R(X)$ is metric separable. In this paper, we show that (a) and (b), together with the condition that $R(X)$ has a countable network, ensure that $X$ has a countable compactification. This includes Hoshina's result as well as the case when $R(X)$ is countable. In addition, our proof is considerably shorter than the one given by Hoshina. Furthermore, we construct examples to show that, in general, the topological properties of $R(X)$ do not determine whether $X$ has a countable compactification.

2. A result.

Theorem. Let $X$ be a Čech-complete, rim-compact space such that $R(X)$ has a countable network. Then $X$ has a countable compactification.

Proof. Since $X$ is rim-compact, $X$ has at least one compactification $Z$ with $\text{ind}(Z - X) < 0$, where $\text{ind}$ denotes small inductive dimension, and since $X$...
is Čech-complete, \( Z - X = \bigcup_{n=1}^{\infty} F_n \), where for each \( n \) in \( N \), the set of positive integers, \( F_n \) is compact [5]. Let \( \{A_n: n \in N\} \) be a network for \( R(X) \). For a fixed \( n \) in \( N \), let \( M = \{m \in N: \overline{A_m} \cap F_n = \emptyset\} \). If \( x \) is a point of \( R(X) \), by regularity of \( Z \), there is an open set \( V \) of \( Z \) and some \( m \) in \( M \) with \( x \in A_m \subseteq V \subseteq \overline{V} \subseteq Z - F_n \). For each \( m \) in \( M \), by normality of \( Z \), there is a cozero set \( G_m \) of \( Z \) with \( A_m \subseteq G_m \subseteq Z - F_n \). Put

\[
E_n = Z - \bigcup_{m \in M} G_m \cup (X - R(X)).
\]

It is readily seen that \( E_n \) is a compact subset of \( Z - X \) such that \( F_n \subseteq E_n \), \( Z - X = \bigcup_{n=1}^{\infty} E_n \) and the complement of \( E_n \) in any compact subset of \( Z - X = (Z - X) \cup R(X) \) is \( \sigma \)-compact. We may further assume that \( E_n \subseteq E_{n+1} \) for each \( n \) in \( N \). Now \( E_{n+1} - E_n \) is a locally compact, \( \sigma \)-compact space with \( \text{ind}(E_{n+1} - E_n) < 0 \). Hence \( E_{n+1} - E_n \) is the union of a countable collection of mutually disjoint compact sets. It follows that \( Z - X = \bigcup_{n=1}^{\infty} B_n \), where, for \( n, m \) in \( N \) with \( n \neq m \), \( B_n \) and \( B_m \) are disjoint compact sets, and \( (Z - X \cup B_n) \cup R(X) = \bigcup_{m,n} C_{n,m} \), where \( C_{n,m} \) is compact for all \( n, m \) in \( N \).

Since \( Z - X \) is Lindelöf and \( \text{ind}(Z - X) < 0 \), then \( \text{dim}(Z - X) < 0 \), where \( \text{dim} \) denotes covering dimension. Hence, if \( E, F \) are disjoint closed sets of \( Z \), there exist disjoint open sets \( G, H \) with \( E \subseteq G, F \subseteq H \) and \( Z - X \subseteq G \cup H \) (see e.g. [1, Proposition 4]). It follows that there are pairs \( G_i, H_i \) of disjoint open sets of \( Z \) with \( (Z - X) \subseteq G_i \cup H_i \), \( i \in N \), and such that \( E \subseteq G_i \) and \( F \subseteq H_i \) for some \( i \) in \( N \) in each of the following cases. Firstly when \( E = B_n \) and \( F = C_n \), secondly when \( E = \overline{A_n} \), \( F = \overline{A_m} \) and \( \overline{A_n} \cap \overline{A_m} = \emptyset \), and thirdly when \( E = \overline{A_n} \), \( F = B_m \) and \( \overline{A_n} \cap B_m = \emptyset \), where \( n, m \) are in \( N \).

We now define an equivalence relation \( \sim \) on \( Z \) as follows. If \( x, y \in B_n \) for some \( n \) in \( N \), then \( x \sim y \) if and only if \( x \) and \( y \) belong to the same member of \( \{G_i, H_i\} \) for each \( i < n \). Otherwise, \( x \sim y \) if and only if \( x = y \). Let \( \pi: Z \to Y \) be the quotient map induced by \( \sim \). The equivalence class \( \pi^{-1}(x) \) of a point \( x \) of \( B_n \) is the closed set \( D_1 \cap \cdots \cap D_n \cap B_n \), where, for \( i < n \), \( D_i \) is the member of \( \{G_i, H_i\} \) which contains \( x \). Hence \( \pi(B_n) \) consists of a finite number of points. Clearly, \( Y \) is a \( T_1 \) compactification of \( X \) with \( Y - X \) countable. To complete the proof, it suffices to show that \( \pi \) is a closed map, since this implies that \( Y \) is normal and therefore Hausdorff.

Let \( S \) be a closed set of \( Z \). Then \( \pi^{-1}(S) = S \cup T \), where \( T = \bigcup_{n=1}^{\infty} T_n \) and \( T_n = \pi^{-1}(S \cap B_n) - S \). Let \( x \) be a limit point of \( T \). It suffices to show that \( x \in S \cup T \), since this implies that \( \pi^{-1}(S) \) is closed and hence \( \pi \) is closed. Since \( T \) is a subset of the closed set \( (Z - X) \cup R(X) \), either \( x \in R(X) \) or, for some \( n \) in \( N \), \( x \in B_n \). We note that, for \( m, k \) in \( N \), since \( \pi(B_m) \) is finite, then \( \pi^{-1}(S \cap B_m) \) is closed, so that if \( x \) is not in \( \bigcup_{m<k} \pi^{-1}(S \cap B_m) \), then \( x \) is a limit point of \( \bigcup_{m>k} T_m \).

We first assume that \( x \in R(X) \). Let \( K = \{k \in N: x \in G_k \cup H_k\} \). For \( k \) in
COMPACTIFICATIONS WITH COUNTABLE REMAINDER

K, write $D_k$ for the element of $\{G_k, H_k\}$ which contains $x$. Now $x$ is a limit point of $\bigcup_{m \geq k} T_m$ and hence there is an element $x_k$ of this set which is contained in $\cap (D_i: i \in K, i < k)$. Let $y_k$ be an element of $S$ with $y_k \sim x_k$. Then, for $i < k$, $y_k \in H_i$ implies $x_k \in H_i$. The infinite subset $\{y_1, y_2, \ldots\}$ of the compact set $S$ has a limit point $y$ in $S$. Suppose $y \neq x$. Either $y \in R(X)$ or $y \in B_n$ for some $n$ in $N$. In the first case, there are open neighbourhoods $U, V$ of $x$ with $\overline{U} \cap \overline{V} = \emptyset$ and $m, n$ in $N$ with $x \in A_m \subset U$ and $y \in A_n \subset V$. Clearly $\overline{A}_m \cap \overline{A}_n = \emptyset$ and hence there is $r$ in $N$ with $\overline{A}_m \subset G_r$ and $\overline{A}_n \subset H_r$. In the second case, let $U$ be a neighbourhood of $x$ with $\overline{U} \cap B_n = \emptyset$ and let $m$ be in $N$ with $x \in A_m \subset U$. Since $\overline{A}_m \cap B_n = \emptyset$, there is an $r$ in $N$ with $\overline{A}_m \subset G_r$ and $B_n \subset H_r$. Now since $y$ is a limit point of $\{y_1, y_2, \ldots\}$, for some $k > r$, $y_k \in H_r$, which implies that $x_k \in H_r$, so that, since $G_r \cap H_r = \emptyset$, $x_k \in G_r = D_r$. This contradicts the fact that $x_k$ is in $\cap (D_i: i \in K, i < k)$ and shows that $x = y$ and hence $x \in S$.

Finally, suppose $x \in B_n$ for some $n \in N$. It remains to show that $x \in \pi^{-1}\pi(S \cap B_n)$. Suppose this is false. For $i \in N$, let $D_i$ be the member of $\{G_i, H_i\}$ which contains $x$. Then $\pi^{-1}\pi(x) = D_1 \cap \ldots \cap D_n \cap B_n$ and $S \cap D_1 \cap \ldots \cap D_n \cap B_n = \emptyset$. The closure $Q$ of $(S - X) \cup D_1 \cap \ldots \cap D_n$ is a compact subset of $(Z - X) \cup R(X)$ which is disjoint from $B_n$. For if $y \in B_n \cap Q$, then $y \in B_n \cap S$, so that for some $j < n$, $y \in D_j$, and if $P_j$ is the member of $\{G_j, H_j\}$ which contains $y$, then $P_j \cap Q = \emptyset$. Thus $Q$ is a compact subspace of $\bigcup_{k=1}^{\infty} C_n \cap k$. Hence there is a finite subset $L$ of $N$ such that $B_n \subset G_i$ for each $i \in L$ and $Q \subset \bigcup (H_i: i \in L)$. Let $k = n + \max L$ and $D = D_1 \cap \ldots \cap D_k$. Since $x \in B_n$, for $i \in L$, $D_i = G_i$. Let $m > k$ and suppose $y \in D \cap T_m$. Then there is $z$ in $S \cap B_m$ with $y \sim z$. For $i < k$, $y$ and $z$ belong to the same element of $\{G_i, H_i\}$. Hence $z \in D$ and it follows that $z \in Q$. Therefore for some $i$ in $L$, $z \in H_i$, which is absurd since $G_i \cap H_i = \emptyset$ and $z \in D \subset D_i = G_i$.

This shows that $x$ is not a limit point of $\bigcup_{m \geq k} T_m$ and since our assumption that $x \in B_n$ and $x \notin \pi^{-1}\pi(S \cap B_n)$ implies that $x$ is not in $\bigcup_{m \leq k} \pi^{-1}\pi(S \cap B_m)$, then $x$ is not a limit point of $T$. This contradiction shows that $x$ must be in $\pi^{-1}\pi(S \cap B_n)$ and completes the proof of the theorem.

3. Some examples. Example 1 shows that there are rim-compact, Čech-complete spaces $X, X_1$, such that, despite $R(X), R(X_1)$ being homeomorphic, $X$ has a countable compactification but not $X_1$. In this example, $R(X)$ is compact. In Example 2, the same pathology is exhibited with $R(X)$ discrete. Hoshina [4] has shown that if a paracompact space $X$ has a countable compactification, then $R(X)$ is Lindelöf. Example 2 shows that, in general, the fact that $X$ has a countable compactification does not imply that $R(X)$ is Lindelöf.

We need the following result of Hoshina [4].

Lemma. If $X$ has a countable compactification and $\mathcal{U}$ is a collection of mutually disjoint open sets of $X$ with $U \cap R(X) \neq \emptyset$ for each $U$ in $\mathcal{U}$, then $\mathcal{U}$ is countable.
Example 1. Let $R$ be the set of real numbers with the usual topology. Then $X = \beta R - N$, where $\beta$ denotes Stone-Cech compactification, has a countable compactification and $R(X) = \beta N - N$ [8, Example 3].

Let $N \cup \{\infty\}$ be the one-point compactification of $N$, $Y = (N \cup \{\infty\}) \times (N \cup \{\infty\}) \times R(X)$ and $X_1 = Y - \{\infty\} \times N \times R(X)$. Since $Y$ is compact and $Y - X_1$ is $\sigma$-compact and zero-dimensional, then $X_1$ is Čech-complete and rim-compact. In addition, $R(X_1) = \{\infty\} \times \{\infty\} \times R(X)$ is homeomorphic with $R(X)$. Let $\mathcal{U}$ be an uncountable collection of mutually disjoint nonempty open sets of $\beta N - N$ [9, p. 77]. For each $U$ in $\mathcal{U}$, let $U^* = (N \cup \{\infty\}) \times (N \cup \{\infty\}) \times U$. Then $\{U^* \cap X_1 : U \in \mathcal{U}\}$ is an uncountable collection of mutually disjoint open sets of $X_1$ with $U^* \cap X_1 \cap R(X_1) \neq \emptyset$ for each $U$ in $\mathcal{U}$. The lemma implies that $X_1$ has no countable compactification.

Example 2. Let $P$ be the set of irrational numbers and $Q$ the set of rational numbers. For each $x$ in $P$, let $\{x_1, x_2, \ldots\}$ be a sequence of rationals converging to $x$ in the usual topology of $R$. A subset $A$ of $R$ is defined to be open if whenever $x \in A \cap P$, then there is $n$ in $N$ with $\{x_n, x_{n+1}, \ldots\} \subset A$. With this topology, $R$ is locally compact and Hausdorff, $Q$ is dense in $R$ and $P$ is a closed subspace of $R$ with discrete topology [7, p. 87]. Let $R \cup \{\infty\}$ be the one-point compactification of $R$, $Y = (N \cup \{\infty\}) \times (R \cup \{\infty\})$ and $X = Y - \{\infty\} \times Q \cup \{\infty\}$. Then $Y$ is a countable compactification of $X_1$, while $R(X) = \{\infty\} \times P$ is not Lindelöf.

Let $Z = (N \cup \{\infty\}) \times Y$ and $X_1 = (Z - \{\infty\} \times Y) \cup \{\infty\} \times \{\infty\} \times P$. Then $X_1$ is Čech-complete and rim-compact, because $Z - X_1$ is $\sigma$-compact and zero-dimensional, and $R(X_1) = \{\infty\} \times \{\infty\} \times P$ is homeomorphic with $R(X)$. However, the lemma implies that the closed subspace $N \times (N \cup \{\infty\}) \times (P \cup \{\infty\}) \cup R(X_1)$ of $X_1$ has no countable compactification, and hence $X_1$ has no countable compactification.

We can obviously choose $X, X_1$ so that $R(X), R(X_1)$ are homeomorphic with the one-point compactification of $P$.

Bibliography

4. ________, Countable-points compactifications for metric spaces, preprint.


**Department of Mathematics, Ahmadu Bello University, Zaria, Nigeria**

*Current address*: Department of Mathematics, University of Nairobi, P. O. Box 30917, Nairobi, Kenya