A NONDEVELOPABLE ČECH-COMPLETE SPACE WITH A POINT-COUNTABLE BASE

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ABSTRACT. An example is presented which is a p-space, in fact a Čech-complete space, which has a point-countable base and is not developable. This answers questions raised by Burke in 1970 and by Burke and Tall in 1972.

1. Introduction and definitions. The notion of Čech-completeness was defined in 1937 [Č]. It is a natural extension of the idea of a complete metric space, and is simply that a space is Čech-complete if it is a $G_{σ}$-set in its Stone-Čech compactification. For the purposes of this note another characterization will be useful. This is essentially the form found by Frolik in 1960 [F], and is the definition we shall use.

DEFINITION 1.1 [Č], [F]. A completely regular $T_2$ space $X$ is Čech-complete if and only if there is a sequence $\langle S_n : n \in \omega \rangle$ of open covers of $X$ such that if $\mathcal{F}$ is a collection of closed subsets of $X$ with finite intersection property and for each $n \in \omega$ there is $F_n \in \mathcal{F}$ and $G_n \in S_n$ with $F_n \subseteq G_n$, then $\bigcap \mathcal{F} \neq \emptyset$.

The class of p-spaces was introduced in 1963 by Arhangel抯kii [A1] as a class containing both the metrizable spaces and the locally compact spaces. The p-spaces, and relatives like the strict p-spaces [A1] and wΔ-spaces [Bo], have been among the most fruitful and thoroughly investigated classes of spaces in general topology. The authors of papers concerning these spaces are far too numerous to recite here.

DEFINITION 1.2 [A1]. A completely regular $T_2$ space $X$ is called a p-space if in the Stone-Čech compactification $βX$ there is a sequence $\langle T_n : n \in \omega \rangle$ of open covers of $X$ such that $\bigcap_{n \in \omega} \overline{st}(x, T_n) \subseteq X$ for each $x \in X$.

Since a locally compact $T_2$ space is open in its Stone-Čech compactification, it is clear that every locally compact $T_2$ space is Čech-complete and that every Čech-complete space is a p-space.

The questions which we will answer in this note were raised by Burke [B] in 1970 and by Burke and Tall [BT] at the Pittsburgh conference in 1972.

QUESTION 1.3 [B]. Is a p-space with a point-countable base a Moore space?

QUESTION 1.4 [BT]. If $X$ is a Čech-complete space having a point-countable base, must $X$ be developable?

Motivation for these questions is contained in the following theorems.

THEOREM 1.5 [CM]. A locally compact $T_2$ space $X$ with a point-countable base is metrizable.
Theorem 1.6 [B], [H₂]. A metacompact p-space with a point-countable base is a Moore space.

Theorem 1.7 [B]. A subparacompact p-space with a point-countable base is a Moore space.

In this note we show that both Questions 1.3 and 1.4 have negative answers by exhibiting a Čech-complete space with a σ-locally countable and σ-disjoint base which is neither perfect nor θ-refinable. It is interesting to compare the covering properties in the example with Burke’s theorems. Having a σ-disjoint base, the example is screenable (thus weakly θ-refinable) which implies metacompact for developable spaces [H₁] and implies subparacompact for perfect spaces [BL]. This shows that there is very little room for improvement in these theorems.

In our set theoretic usage, we will follow the custom that cardinal numbers are initial ordinals, and we use c to denote the cardinality of the reals, i.e. c = 2ω.

2. The example. Before launching into the technical details of the construction, we feel it will be helpful to give a brief intuitive description of the example. We begin with a zero-dimensional, nonseparable metric space M, the so-called “Baire space of weight c”. To this we attach a closed discrete set F of cardinality c. Neighborhoods of points of F will be tails of carefully selected discrete sequences of metric balls in M. The critical property of the sequences, in proving Čech-completeness, is that no two terms of a sequence can intersect the same term of another sequence.

Example 2.1 There is a zero-dimensional T₂ space Z which is Čech-complete, has a σ-locally countable and σ-disjoint base (hence, a point-countable base), but is not developable.

Proof. Let D be a discrete space of cardinality c, let M = Πₖ∈ωDₖ, where Dₖ = D for each k ∈ ω, and let F = {α: α < c}. For each point x ∈ Πₖ=₀Dₖ, we denote by [x] the basic open set in M which is given by {z: πᵢz = πᵢx for i < k}. Let ℳₖ = {x: x ∈ Πₖ=₀Dₖ} and let {Sₐ: α < c} be a well ordering of all countable subsets of Σₖ∈ωℳₖ such that each Sₐ is contained in some ℳₖ(α), and the projection of Sₐ into D₀ is one-to-one. We define, by induction on α, sequences sₐ: ω → Σₐ and Bₐ: ω → Σₖ∈ωℳₖ such that the following are true for each α < c:

1. If i ≠ j, sₐ(i) and sₐ(j) are not in the same element of Sₐ.
2. sₐ(n) ∈ Bₐ(n).
3. Bₐ(n) ⊆ i>n+k(α) ℳᵢ.
4. If α < β, then (Σₖ∈ωBₐ(n)) ∩ {sₐ(k): k ∈ ω} = ∅, and
   \[ \left( \bigcup_{i > k(β) + 1} \left( \bigcup_{n ∈ ω} Bₐ(n) \right) \right) \cap \left( \bigcup_{n ∈ ω} Bₐ(n) \right) = \emptyset. \]
5. If Bₐ(n) = {x₁, x₂, . . . , xₖ(β)}, then [x₁, x₂, . . . , xₖ(β)+₁] ∩ Bₐ(n) = ∅ whenever α < β and Bₐ(n) ⊆ Σₖ∈ωℳₖ.

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The space $Z = M \cup F$, and $M$ is an open subspace. For $\alpha \in F$, we define $U_n(\alpha) = \{\alpha\} \cup (\bigcup_{k > n} F_\alpha(k))$. The collection $\{U_n(\alpha): n \in \omega\}$ will be an open neighborhood base at $\alpha$. It is easy to see that this topology on $Z$ is a zero-dimensional $T_2$ space, hence $Z$ is completely regular.

We now show that $Z$ is Čech-complete. For $x \in M$, we let $U_n(x) = [x_1, x_2, \ldots, x_n] \in \mathfrak{p}_n$, for each $n \in \omega$. Note that with the usual metric on $M$, $U_n(x) = B(x, 2^{-n})$. For $n \in \omega$ let $\delta_n = \{U_n(z): z \in Z\}$. It is clear that $\delta_n$ is an open cover of $Z$ for each $n \in \omega$. Suppose $\mathfrak{F}$ is a collection of closed subsets of $Z$ with finite intersection property and for each $n \in \omega$ there is $F_n \in \mathfrak{F}$ and $G_n \in \delta_n$ with $F_n \subseteq G_n$. We let $H_n = \cap_i \leq n F_i$ for each $n \in \omega$.

First, if there exists $k \in \omega$ with $G_k = \{a\}$ for some $a \in F$, then $H_n \cap G_k = \emptyset$ for every $n \in \omega$. Suppose $\mathcal{S}$ is an open cover of $Z$ for each $n \in \omega$. Suppose $\mathcal{S}$ is a collection of closed subsets of $Z$ with finite intersection property and for each $n \in \omega$ there is $F_n \in \mathfrak{F}$ and $G_n \in \delta_n$ with $F_n \subseteq G_n$. We let $H_n = \cap_i \leq n F_i$ for each $n \in \omega$. First, if there exists $k \in \omega$ with $G_k = \{a\}$ for some $a \in F$, then $H_n \cap G_k = \emptyset$ for every $n \in \omega$. Suppose $\mathcal{S}$ is a collection of closed subsets of $Z$ with finite intersection property and for each $n \in \omega$ there is $F_n \in \mathfrak{F}$ and $G_n \in \delta_n$ with $F_n \subseteq G_n$. We let $H_n = \cap_i \leq n F_i$ for each $n \in \omega$. First, if there exists $k \in \omega$ with $G_k = \{a\}$ for some $a \in F$, then $H_n \cap G_k = \emptyset$ for every $n \in \omega$.

This example was created by Gruenhage ([DGN, Example 3.3]) as an example of a space with a $\sigma$-locally countable base which is not developable.

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a strict $p$-space, is not a $w\Delta$-space, is not countably metacompact, is not collectionwise Hausdorff.

REFERENCES


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