ON THE CHERN CLASSES
AND THE EULER CHARACTERISTIC
FOR NONSINGULAR COMPLETE INTERSECTIONS

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ABSTRACT. It is shown that Hirzebruch's result on the Chern classes of a
complete intersection of nonsingular hypersurfaces in general position in
$\mathbb{P}_N(\mathbb{C})$, is also valid for any nonsingular complete intersection. Then rela-
tions between Euler characteristic, class and Milnor number are pointed out.

1. Introduction. Let $\mathbb{P}_N(\mathbb{C})$ be an $N$-dimensional complex projective space.
An $n$-dimensional algebraic subvariety $V$ of $\mathbb{P}_N(\mathbb{C})$ is called a complete
intersection if the homogeneous ideal of $V$ in $\mathbb{C}[z_0, \ldots, z_N]$ is generated by
$p = N - n$ homogeneous polynomials, $f_1, \ldots, f_p$. So, if $V_i = \text{loc}[f_i]$, we have
$V = V_1 \cap \ldots \cap V_p$.

It is well known that if the $V_i$ are nonsingular and are in general position,
then $V$ is nonsingular too; in this case the Chern classes of $V$ can be obtained
in terms of the degrees of $V_1, \ldots, V_p$ (see [11]). Nevertheless, examples are
easily found in which, though $V$ is nonsingular, it is not possible to find
generators $f_i$ of the homogeneous ideal of $V$ that would give nonsingular $V_i$,
$1 < i < p$. This is, for example, the case of the sextic

$$z_0^3 + z_1^3 + z_2^3 = 0, \quad z_1^2 + z_2^2 + z_3^2 = 0$$

in the space $\mathbb{P}_3(\mathbb{C})$.

In §2 we prove that it is not necessary to assume that the hypersurfaces $V_i$
are nonsingular and in general position, since the expression of the Chern
classes of $V$ in terms of the degrees of the $V_i$ is valid even if we only assume
that $V$ is nonsingular.

In §3 we give an elementary proof for the formula giving the Euler
characteristic of a nonsingular complete intersection that does not use its
relation with the top Chern class, while in §4 we give two applications of this
formula, obtaining the class of a nonsingular complete intersection and the
Milnor number of a germ of a homogeneous complete intersection with an
isolated singularity.

2. The Chern classes. The result for which we are aiming will be obtained
from the following.
Lemma. If $V$ is an $n$-dimensional nonsingular complete intersection in $\mathbb{P}_n(\mathbb{C})$, then there is a system of generators $f_1, \ldots, f_p$ of the homogeneous ideal of $V$, with $p = N - n$, so that the varieties

$$W_i = \text{loc}\left[ f_1, \ldots, f_i \right], \quad 1 \leq i \leq p,$$

are nonsingular.

Proof. Let $g_1, \ldots, g_p$ be an arbitrary system of generators of $\text{id}(V)$ and suppose that $d_1 > d_2 > \cdots > d_p$, where $d_i = \deg g_i$, $1 \leq i \leq p$. Then, if we consider the linear system of divisors of degree $d_i$ defined by

$$\lambda_1 g_1 + \sum_{1 \leq j \leq p} \lambda_{ij} z_j^{d_i - d_i} g_i,$$

we can deduce, from the parametric transversality theorem in [9] (essentially the Sard theorem), that there are constants $\lambda_{ij}$ so that if

$$f_1 = g_1 + \sum_{1 \leq j \leq p} \lambda_{ij} z_j^{d_i - d_i} g_i$$

then $W_1 = \text{loc}[f_1]$ is nonsingular. Suppose now that we have already found $f_1, \ldots, f_k$, where $k > 1$, so that $[f_1, \ldots, f_k, g_{k+1}, \ldots, g_p] = \text{id}(V)$ and $W_j = \text{loc}[f_1, \ldots, f_j]$ is nonsingular if $1 \leq j < k$. Considering the linear system on $W_k$ defined by

$$\mu_1 g_{k+1} + \sum_{k \leq i \leq p} \mu_{ij} z_j^{d_k - d_k} g_i$$

we deduce, again from the parametric transversality theorem, that there are constants $\mu_{ij}$ so that if

$$f_{k+1} = g_{k+1} + \sum_{k \leq i \leq p} \mu_{ij} z_j^{d_k - d_k} g_i$$

then $W_{k+1} = \text{loc}[f_1, \ldots, f_{k+1}]$ is nonsingular and obviously $[f_1, \ldots, f_{k+1}, g_{k+2}, \ldots, g_p] = \text{id}(V)$.

Let $H^d$ be a line bundle over $\mathbb{P}_n(\mathbb{C})$ defined by the linear system of hypersurfaces of degree $d > 1$ in $\mathbb{P}_n(\mathbb{C})$. It is well known that $H^d$ has $c_1(H^d) = dh \in H^2(\mathbb{P}_n(\mathbb{C}), \mathbb{Z})$, where $h$ is a generator of $H^2(\mathbb{P}_n(\mathbb{C}), \mathbb{Z})$.

Let $V$ be a nonsingular complete intersection in $\mathbb{P}_n(\mathbb{C})$. We shall say that $V$ has degrees $(d_1, \ldots, d_p)$ if $\text{id}(V) = [f_1, \ldots, f_p]$ and $d_i = \deg f_i$, $1 \leq i \leq p$. If $j : V \to \mathbb{P}_n(\mathbb{C})$ is the embedding of $V$ in $\mathbb{P}_n(\mathbb{C})$, we shall write $h$ for $j^*h$.

Theorem 1. Let $V$ be a nonsingular complete intersection in $\mathbb{P}_n(\mathbb{C})$ of degrees $(d_1, \ldots, d_p)$, the total Chern class of $V$ is given by

$$c(V) = (1 + h)^{N+1}(1 + d_1h)^{-1} \cdots (1 + d_p h)^{-1}.$$

Proof. The proof follows from the previous lemma and from [11, 4.8.1] by induction, as in [11].

Remark. Applying the previous lemma, the other results of [10] or [11] about complete intersections of nonsingular hypersurfaces in general position are likewise valid for any nonsingular complete intersection.
3. The Euler characteristic. The following well-known lemma plays an important role in the development of this section.

**Lemma.** Let $V$ and $W$ be two $n$-dimensional nonsingular complete intersections in $\mathbb{P}_n(\mathbb{C})$ of the same degrees $(d_1, \ldots, d_p)$. Then $V$ and $W$ are diffeomorphic manifolds.

Since all the $n$-dimensional nonsingular complete intersections in $\mathbb{P}_n(\mathbb{C})$ of degrees $(d_1, \ldots, d_p)$ have, according to the above lemma, the same Euler characteristic, we will write it as $\chi(d_1, \ldots, d_p; n)$; hence we can choose a particularly simple complete intersection, so that:

**Lemma.** If $d_1, \ldots, d_p$ are integers $> 1$, for almost any $(a_{ij})$, $1 < j < p$, $j - 1 < i < N$, $a_{ij} \in \mathbb{C}$, there is a nonsingular complete intersection $V$ in $\mathbb{P}_N(\mathbb{C})$ of degrees $(d_1, \ldots, d_p)$, with ideal

$$\text{id}(V) = \left\{ \sum_{i=1}^{N} a_{ij} z_i^q ; 1 < j < p \right\}$$

and such that $V \cap \{ z_0 = 0 \}$ is nonsingular.

**Proof.** It is again a simple application of the parametric transversality theorem, since, if we consider the mappings

$$F_1: \mathbb{C}^r \to C^\infty(\mathbb{C}^{N+1} - \{0\}, \mathbb{C}^r),$$

$$(a_{ij}) \to f(z_0, \ldots, z_N) = \left( \sum_{i=0}^{N} a_{i1} z_i^d_1, \ldots, \sum_{i=p-1}^{N} a_{ip} z_i^d_p \right),$$

and

$$F_2: \mathbb{C}^r \to C^\infty(\mathbb{C}^{N} - \{0\}, \mathbb{C}^r),$$

$$(a_{ij}) \to g(z_1, \ldots, z_N) = \left( \sum_{i=1}^{N} a_{i1} z_i^d_1, \ldots, \sum_{i=p-1}^{N} a_{ip} z_i^d_p \right),$$

we immediately verify that $\phi(F_1, \{0\})$ and $\phi(F_2, \{0\})$ are open dense sets so that their intersection will also be dense.

**Theorem 2.** With the same notations as above,

$$\chi(d_1, \ldots, d_p; n) = d_1 d_2 \cdots d_p \left\{ \sum_{i=0}^{n} (-1)^{n-i} \binom{N+1}{i} A_p^{n-i} \right\}$$

where

$$A_p^k = \sum_{|a|=k} d_1^{a_1} d_2^{a_2} \cdots d_p^{a_p}.$$
according to the Lefschetz duality theorem, we have
\[ \chi(W - V_0) = \chi(W, V_0) = \chi(W) - \chi(V_0) \]
and also
\[ \chi(V) = \chi(V_0) + \chi(V_0); \]
therefore
\[ \chi(V) = d_1 \chi(W) - (d_1 - 1) \chi(V_0). \]
We thus obtain the induction formula
\[ \chi(d_1, \ldots, d_p; n) = d_1 \chi(d_2, \ldots, d_p; n) - (d_1 - 1) \chi(d_1, \ldots, d_p; n - 1) \]
from which the theorem follows easily, starting from the initial values
\[ \chi(\mathbb{P}_n(C)) = n + 1 \text{ and } \chi(d_1, \ldots, d_p; 0) = d_1 d_2 \cdots d_p. \]

**Remark.** A noteworthy consequence of the previous theorem is that the results contained in [2], [3], [4] and [5] are also valid for any nonsingular complete intersection.

4. Some applications. We can now apply the anterior result in order to obtain:

**Theorem 3 (Compare with [14]).** If \( V \) is an \( n \)-dimensional nonsingular complete intersection in \( \mathbb{P}_n(C) \) of degrees \( (d_1, \ldots, d_p) \), its class is given by
\[ \rho_n(d_1, \ldots, d_p; n) = d_1 d_2 \cdots d_p \left\{ \sum_{i=0}^{n} (-1)^i \binom{N-1}{i} A_{p-i} \right\}. \]

**Proof.** It is known (see [12]) that the dual variety \( V^* \) of a nonsingular complete intersection \( V \) in \( \mathbb{P}_n(C) \) is a hypersurface in the dual projective space \( \mathbb{P}_N^*(C) \) of the hyperplanes; therefore the class of \( V \), i.e. the degree of \( V^* \), will be obtained as the intersection number of \( V^* \) with a general line of \( \mathbb{P}_N^*(C) \). But if we interpret this line as a pencil of hyperplanes in \( \mathbb{P}_n(C) \) and apply the Zeuthen-Segre formula in [1], we have
\[ \chi(V) = 2 \chi(V \cap H_0) - \chi(V \cap H_0 \cap H_1) + (-1)^n \rho_n(V) \]
where \( H_0 \) and \( H_1 \) are two generic hyperplanes of the pencil, from which it follows:
\[ \rho_n(V) = (-1)^n \left\{ \chi(d_1, \ldots, d_p; n) - 2 \chi(d_1, \ldots, d_p; n - 1) \right. \]
\[ \left. + \chi(d_1, \ldots, d_p; n - 2) \right\} \]
and applying Theorem 2, we obtain the expression for which we were aiming.

Finally, we apply Theorem 2 to obtain the Milnor number of the germs of homogeneous complete intersection with an isolated singularity.

Let \((X, 0)\) be an \( n \)-dimensional germ of complete intersection in \( \mathbb{C}^N \), defined in a neighborhood \( U \) of 0 by the functions \( f_1, \ldots, f_p \); we shall say that \((X, 0)\) is an homogeneous complete intersection of degrees \( (d_1, \ldots, d_p) \) if \( f_i \) is an homogeneous polynomial of degree \( d_i \), \( 1 \leq i \leq p \). From this it follows immediately that in this case we can take all \( \mathbb{C}^N \) as \( U \). We assume now that
(X, 0) has an isolated singularity, and we denote by f: C^N \to C^p the mapping with components f_1, \ldots, f_p; the affine manifold f^{-1}(y), where y is a regular value of f, is called the Milnor fibre of (X, 0) (see [13] and [8]). From [8] we thus know that f^{-1}(y) has the homotopy type of a bouquet of n-dimensional spheres; the number of spheres in this bouquet is called the Milnor number of (X, 0), \mu(X, 0). The following result is originally stated in [7] (see also [6]).

**Theorem 4.** If (X, 0) is an n-dimensional homogeneous complete intersection in C^N, of degrees (d_1, \ldots, d_p) with an isolated singularity, we have

\[ \mu(X, 0) = (-1)^{n+1} + d_1 d_2 \cdots d_p \left\{ \sum_{i=0}^n (-1)^i \binom{N}{i} A_p^{n-i} \right\}. \]

**Proof.** Let F = f^{-1}(y) be the Milnor fibre. Using a change of coordinates in C^p, if necessary, we may assume that y = (1, 1, \ldots, 1). Since f is homogeneous, we may define the projective variety V of P_n(C) as

\[ V = \{ z \in P_n(C); f(z_1, \ldots, z_N) - z_0^d = 0, 1 < i < p \} \]

so V is a nonsingular complete intersection of degrees (d_1, \ldots, d_p) and, if V_0 = V \cap \{ z_0 = 0 \}, we shall get F = V - V_0. But since V_0 is a complete nonsingular intersection in P_{n-1}(C), of degrees (d_1, \ldots, d_p), we shall have

\[ \chi(F) = \chi(V, V_0) = \chi(V) - \chi(V_0) \]

i.e.

\[ 1 + (-1)^n \mu = \chi(F) = \chi(d_1, \ldots, d_p; n) - \chi(d_1, \ldots, d_p; n-1). \]

Now applying Theorem 2, we have the formula we sought to obtain.

**Bibliography**


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