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ON THE CHERN CLASSES AND THE EULER CHARACTERISTIC FOR NONSINGULAR COMPLETE INTERSECTIONS

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ABSTRACT. It is shown that Hirzebruch's result on the Chern classes of a complete intersection of nonsingular hypersurfaces in general position in $\mathbf{P}_N(\mathbf{C})$, is also valid for any nonsingular complete intersection. Then relations between Euler characteristic, class and Milnor number are pointed out.

1. Introduction. Let $\mathbf{P}_N(\mathbf{C})$ be an N-dimensional complex projective space. An n-dimensional algebraic subvariety V of $\mathbf{P}_N(\mathbf{C})$ is called a complete intersection if the homogeneous ideal of V in $\mathbf{C}[z_0, \ldots, z_N]$ is generated by $p = N - n$ homogeneous polynomials, $f_1, \ldots, f_p$. So, if $V_i = \operatorname{loc}[f_i]$, we have $V = V_1 \cap \ldots \cap V_p$.

It is well known that if the V_i are nonsingular and are in general position, then V is nonsingular too; in this case the Chern classes of V can be obtained in terms of the degrees of $V_1, \ldots, V_p$ (see **[11]**). Nevertheless, examples are easily found in which, though V is nonsingular, it is not possible to find generators f_i of the homogeneous ideal of V that would give nonsingular V_i, $1 \leqslant i \leqslant p$. This is, for example, the case of the sextic

$$z_0^3 + z_1^3 + z_2^3 = 0, \qquad z_1^2 + z_2^2 + z_3^2 = 0$$

in the space $\mathbf{P}_3(\mathbf{C})$.

In §2 we prove that it is not necessary to assume that the hypersurfaces V_i are nonsingular and in general position, since the expression of the Chern classes of V in terms of the degrees of the V_i is valid even if we only assume that V is nonsingular.

In §3 we give an elementary proof for the formula giving the Euler characteristic of a nonsingular complete intersection that does not use its relation with the top Chern class, while in §4 we give two applications of this formula, obtaining the class of a nonsingular complete intersection and the Milnor number of a germ of a homogeneous complete intersection with an isolated singularity.

2. The Chern classes. The result for which we are aiming will be obtained from the following.

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LEMMA. *If V is an n-dimensional nonsingular complete intersection in $\mathbf{P}_N(\mathbf{C})$, then there is a system of generators $f_1, \ldots, f_p$ of the homogeneous ideal of V, with $p = N - n$, so that the varieties*

$$W_i = \operatorname{loc}[f_1, \ldots, f_i], \qquad 1 \leqslant i \leqslant p,$$

are nonsingular.

PROOF. Let $g_1, \ldots, g_p$ be an arbitrary system of generators of $\operatorname{id}(V)$ and suppose that $d_1 \geqslant d_2 \geqslant \cdots \geqslant d_p$, where $d_i = \deg g_i$, $1 \leqslant i \leqslant p$. Then, if we consider the linear system of divisors of degree d_1 defined by

$$\lambda_1 g_1 + \sum_{\substack{1 < i \leqslant p \\ 0 \leqslant j \leqslant N}} \lambda_{ij} z_j^{d_1 - d_i} g_i,$$

we can deduce, from the parametric transversality theorem in **[9]** (essentially the Sard theorem), that there are constants λ_{ij} so that if

$$f_1 = g_1 + \sum \lambda_{ij} z_j^{d_1 - d_i} g_i$$

then $W_1 = \operatorname{loc}[f_1]$ is nonsingular. Suppose now that we have already found $f_1, \ldots, f_k$, where $k \geqslant 1$, so that $[f_1, \ldots, f_k, g_{k+1}, \ldots, g_p] = \operatorname{id}(V)$ and $W_j = \operatorname{loc}[f_1, \ldots, f_j]$ is nonsingular if $1 \leqslant j \leqslant k$. Considering the linear system on W_k defined by

$$\mu_1 g_{k+1} + \sum_{\substack{k < i \leqslant p \\ 0 \leqslant j \leqslant N}} \mu_{ij} z_j^{d_{k+1} - d_i} g_i$$

we deduce, again from the parametric transversality theorem, that there are constants μ_{ij} so that if

$$f_{k+1} = g_{k+1} + \sum \mu_{ij} z_j^{d_{k+1} - d_i} g_i$$

then $W_{k+1} = \operatorname{loc}[f_1, \ldots, f_{k+1}]$ is nonsingular and obviously $[f_1, \ldots, f_{k+1}, g_{k+2}, \ldots, g_p] = \operatorname{id}(V)$.

Let H^d be a line bundle over $\mathbf{P}_N(\mathbf{C})$ defined by the linear system of hypersurfaces of degree $d \geqslant 1$ in $\mathbf{P}_N(\mathbf{C})$. It is well known that H^d has $c_1(H^d) = dh \in H^2(\mathbf{P}_N(\mathbf{C}), \mathbf{Z})$, where h is a generator of $H^2(\mathbf{P}_N(\mathbf{C}), \mathbf{Z})$.

Let V be a nonsingular complete intersection in $\mathbf{P}_N(\mathbf{C})$. We shall say that V has degrees $(d_1, \ldots, d_p)$ if $\operatorname{id}(V) = [f_1, \ldots, f_p]$ and $d_i = \deg f_i$, $1 \leqslant i \leqslant p$. If $j\colon V \to \mathbf{P}_N(\mathbf{C})$ is the embedding of V in $\mathbf{P}_N(\mathbf{C})$, we shall write $\tilde{h}$ for $j^* h$.

THEOREM 1. *Let V be a nonsingular complete intersection in $\mathbf{P}_N(\mathbf{C})$ of degrees $(d_1, \ldots, d_p)$, the total Chern class of V is given by*

$$c(V) = (1 + \tilde{h})^{N+1} (1 + d_1 \tilde{h})^{-1} \cdots (1 + d_p \tilde{h})^{-1}.$$

PROOF. The proof follows from the previous lemma and from **[11**, 4.8.1**]** by induction, as in **[11]**.

REMARK. Applying the previous lemma, the other results of **[10]** or **[11]** about complete intersections of nonsingular hypersurfaces in general position are likewise valid for any nonsingular complete intersection.

3. The Euler characteristic. The following well-known lemma plays an important role in the development of this section.

LEMMA. *Let V and W be two n-dimensional nonsingular complete intersections in $\mathbf{P}_N(\mathbf{C})$ of the same degrees $(d_1, \ldots, d_p)$. Then V and W are diffeomorphic manifolds.*

Since all the n-dimensional nonsingular complete intersections in $\mathbf{P}_N(\mathbf{C})$ of degrees $(d_1, \ldots, d_p)$ have, according to the above lemma, the same Euler characteristic, we will write it as $\chi(d_1, \ldots, d_p; n)$; hence we can choose a particularly simple complete intersection, so that:

LEMMA. *If $d_1, \ldots, d_p$ are integers $\geqslant 1$, for almost any (a_{ij}), $1 \leqslant j \leqslant p$, $j - 1 \leqslant i \leqslant N$, $a_{ij} \in \mathbf{C}$, there is a nonsingular complete intersection V in $\mathbf{P}_N(\mathbf{C})$ of degrees $(d_1, \ldots, d_p)$, with ideal*

$$\operatorname{id}(V) = \left[\left\{ \sum_{i=j-1}^{N} a_{ij} z_i^{d_j};\ 1 \leqslant j \leqslant p \right\}\right]$$

and such that $V \cap \{z_0 = 0\}$ is nonsingular.

PROOF. It is again a simple application of the parametric transversality theorem, since, if we consider the mappings

$$F_1: \mathbf{C}^\nu \to C^\infty(\mathbf{C}^{N+1} - \{0\}, \mathbf{C}^p),$$

$$(a_{ij}) \to f(z_0, \ldots, z_N) = \left(\sum_{i=0}^{N} a_{i1} z_i^{d_1}, \ldots, \sum_{i=p-1}^{N} a_{ip} z_i^{d_p} \right),$$

and

$$F_2: \mathbf{C}^\nu \to C^\infty(\mathbf{C}^{N} - \{0\}, \mathbf{C}^p),$$

$$(a_{ij}) \to g(z_1, \ldots, z_N) = \left(\sum_{i=1}^{N} a_{i1} z_i^{d_1}, \ldots, \sum_{i=p-1}^{N} a_{ip} z_i^{d_p} \right),$$

we immediately verify that $\pitchfork(F_1, \{0\})$ and $\pitchfork(F_2, \{0\})$ are open dense sets so that their intersection will also be dense.

THEOREM 2. *With the same notations as above,*

$$\chi(d_1, \ldots, d_p; n) = d_1 d_2 \cdots d_p \left\{ \sum_{i=0}^{n} (-1)^{n-i} \binom{N+1}{i} A_p^{n-i} \right\}$$

where

$$A_p^k = \sum_{|\alpha| = k} d_1^{\alpha_1} d_2^{\alpha_2} \cdots d_p^{\alpha_p}.$$

PROOF. Let $W = \{z \in \mathbf{P}_{N-1};\ \sum_{i=j-1}^{N} a_{ij} z_i^{d_j} = 0, 2 \leqslant j \leqslant p\}$, $V_0 = V \cap \{z_0 = 0\}$ and $V_a = V - V_0$. The map $f: V_a \to W - V_0$ defined by

$$f([z_0, \ldots, z_N]) = [z_1, \ldots, z_N]$$

is a d_1-sheeted covering projection, therefore $\chi(V_a) = d_1 \chi(W - V_0)$. But according to the Lefschetz duality theorem, we have

$$\chi(W - V_0) = \chi(W, V_0) = \chi(W) - \chi(V_0)$$

and also

$$\chi(V) = \chi(V_a) + \chi(V_0);$$

therefore

$$\chi(V) = d_1\chi(W) - (d_1 - 1)\chi(V_0).$$

We thus obtain the induction formula

$$\chi(d_1, \ldots, d_p; n) = d_1\chi(d_2, \ldots, d_p; n) - (d_1 - 1)\chi(d_1, \ldots, d_p; n - 1)$$

from which the theorem follows easily, starting from the initial values $\chi(\mathbf{P}_n(\mathbf{C})) = n + 1$ and $\chi(d_1, \ldots, d_p; 0) = d_1 d_2 \cdots d_p$.

REMARK. A noteworthy consequence of the previous theorem is that the results contained in **[2]**, **[3]**, **[4]** and **[5]** are also valid for any nonsingular complete intersection.

4. Some applications. We can now apply the anterior result in order to obtain:

THEOREM 3 (Compare with **[14]**). *If V is an n-dimensional nonsingular complete intersection in $\mathbf{P}_N(\mathbf{C})$ of degrees $(d_1, \ldots, d_p)$, its class is given by*

$$\rho_n(d_1, \ldots, d_p; n) = d_1 d_2 \cdots d_p \left\{ \sum_{i=0}^{n} (-1)^i \binom{N-1}{i} A_p^{n-i} \right\}.$$

PROOF. It is known (see **[12]**) that the dual variety V^* of a nonsingular complete intersection V in $\mathbf{P}_N(\mathbf{C})$ is a hypersurface in the dual projective space $\mathbf{P}_N^*(\mathbf{C})$ of the hyperplanes; therefore the class of V, i.e. the degree of V^*, will be obtained as the intersection number of V^* with a general line of $\mathbf{P}_N^*(\mathbf{C})$. But if we interpret this line as a pencil of hyperplanes in $\mathbf{P}_N(\mathbf{C})$ and apply the Zeuthen-Segre formula in **[1]**, we have

$$\chi(V) = 2\chi(V \cap H_0) - \chi(V \cap H_0 \cap H_1) + (-1)^n \rho_n(V)$$

where H_0 and H_1 are two generic hyperplanes of the pencil, from which it follows:

$$\rho_n(V) = (-1)^n \{\chi(d_1, \ldots, d_p; n) - 2\chi(d_1, \ldots, d_p; n - 1) + \chi(d_1, \ldots, d_p; n - 2)\}$$

and applying Theorem 2, we obtain the expression for which we were aiming.

Finally, we apply Theorem 2 to obtain the Milnor number of the germs of homogeneous complete intersection with an isolated singularity.

Let $(X, 0)$ be an n-dimensional germ of complete intersection in $\mathbf{C}^N$, defined in a neighborhood U of 0 by the functions $f_1, \ldots, f_p$; we shall say that $(X, 0)$ is an homogeneous complete intersection of degrees $(d_1, \ldots, d_p)$ if f_i is an homogeneous polynomial of degree d_i, $1 \leqslant i \leqslant p$. From this it follows immediately that in this case we can take all $\mathbf{C}^N$ as U. We assume now that

$(X, 0)$ has an isolated singularity, and we denote by $f\colon \mathbf{C}^N \to \mathbf{C}^p$ the mapping with components $f_1, \ldots, f_p$; the affine manifold $f^{-1}(y)$, where y is a regular value of f, is called the Milnor fibre of $(X, 0)$ (see **[13]** and **[8]**). From **[8]** we thus know that $f^{-1}(y)$ has the homotopy type of a bouquet of n-dimensional spheres; the number of spheres in this bouquet is called the Milnor number of $(X, 0)$, $\mu(X, 0)$. The following result is originally stated in **[7]** (see also **[6]**).

THEOREM 4. *If $(X, 0)$ is an n-dimensional homogeneous complete intersection in $\mathbf{C}^N$, of degrees $(d_1, \ldots, d_p)$ with an isolated singularity, we have*

$$\mu(X, 0) = (-1)^{n+1} + d_1 d_2 \cdots d_p \left\{ \sum_{i=0}^{n} (-1)^i \binom{N}{i} A_p^{n-i} \right\}.$$

PROOF. Let $F = f^{-1}(y)$ be the Milnor fibre. Using a change of coordinates in $\mathbf{C}^p$, if necessary, we may assume that $y = (1, 1, \ldots, 1)$. Since f is homogeneous, we may define the projective variety V of $\mathbf{P}_N(\mathbf{C})$ as

$$V = \left\{ z \in \mathbf{P}_N(\mathbf{C}); f_i(z_1, \ldots, z_N) - z_0^{d_i} = 0, 1 \leqslant i \leqslant p \right\}$$

so V is a nonsingular complete intersection of degrees $(d_1, \ldots, d_p)$ and, if $V_0 = V \cap \{z_0 = 0\}$, we shall get $F = V - V_0$. But since V_0 is a complete nonsingular intersection in $\mathbf{P}_{N-1}(\mathbf{C})$, of degrees $(d_1, \ldots, d_p)$, we shall have

$$\chi(F) = \chi(V, V_0) = \chi(V) - \chi(V_0)$$

i.e.

$$1 + (-1)^n \mu = \chi(F) = \chi(d_1, \ldots, d_p; n) - \chi(d_1, \ldots, d_p; n-1).$$

Now applying Theorem 2, we have the formula we sought to obtain.

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