ON THE CHERN CLASSES
AND THE EULER CHARACTERISTIC
FOR NONSINGULAR COMPLETE INTERSECTIONS

VICENTE NAVARRO AZNAR

Abstract. It is shown that Hirzebruch's result on the Chern classes of a complete intersection of nonsingular hypersurfaces in general position in \( \mathbb{P}_N(\mathbb{C}) \), is also valid for any nonsingular complete intersection. Then relations between Euler characteristic, class and Milnor number are pointed out.

1. Introduction. Let \( \mathbb{P}_N(\mathbb{C}) \) be an \( N \)-dimensional complex projective space. An \( n \)-dimensional algebraic subvariety \( V \) of \( \mathbb{P}_N(\mathbb{C}) \) is called a complete intersection if the homogeneous ideal of \( V \) in \( \mathbb{C}[z_0, \ldots, z_N] \) is generated by \( p = N - n \) homogeneous polynomials, \( f_1, \ldots, f_p \). So, if \( V_i = \text{loc}[f_i] \), we have \( V = V_1 \cap \cdots \cap V_p \).

It is well known that if the \( V_i \) are nonsingular and are in general position, then \( V \) is nonsingular too; in this case the Chern classes of \( V \) can be obtained in terms of the degrees of \( V_1, \ldots, V_p \) (see [11]). Nevertheless, examples are easily found in which, though \( V \) is nonsingular, it is not possible to find generators \( f_i \) of the homogeneous ideal of \( V \) that would give nonsingular \( V_i \), \( 1 < i < p \). This is, for example, the case of the sextic

\[
z_0^3 + z_1^3 + z_2^3 = 0, \quad z_1^2 + z_2^2 + z_3^2 = 0
\]

in the space \( \mathbb{P}_3(\mathbb{C}) \).

In §2 we prove that it is not necessary to assume that the hypersurfaces \( V_i \) are nonsingular and in general position, since the expression of the Chern classes of \( V \) in terms of the degrees of the \( V_i \) is valid even if we only assume that \( V \) is nonsingular.

In §3 we give an elementary proof for the formula giving the Euler characteristic of a nonsingular complete intersection that does not use its relation with the top Chern class, while in §4 we give two applications of this formula, obtaining the class of a nonsingular complete intersection and the Milnor number of a germ of a homogeneous complete intersection with an isolated singularity.

2. The Chern classes. The result for which we are aiming will be obtained from the following.
Lemma. If \( V \) is an \( n \)-dimensional nonsingular complete intersection in \( \mathbb{P}_N(\mathbb{C}) \), then there is a system of generators \( f_1, \ldots, f_p \) of the homogeneous ideal of \( V \), with \( p = N - n \), so that the varieties
\[
W_i = \text{loc}(f_1, \ldots, f_i), \quad 1 \leq i \leq p,
\]
are nonsingular.

Proof. Let \( g_1, \ldots, g_p \) be an arbitrary system of generators of \( \text{id}(V) \) and suppose that \( d_1 > d_2 > \cdots > d_p \), where \( d_i = \deg g_i, 1 \leq i \leq p \). Then, if we consider the linear system of divisors of degree \( d_i \) defined by
\[
\lambda_1 g_1 + \sum_{1 < i < p} \lambda_{ij} z_j^{d_i - d_i} g_i,
\]
we can deduce, from the parametric transversality theorem in [9] (essentially the Sard theorem), that there are constants \( \lambda_{ij} \) so that if
\[
f_1 = g_1 + \sum_{1 < i < p} \lambda_{ij} z_j^{d_i - d_i} g_i
\]
then \( W_1 = \text{loc}(f_1) \) is nonsingular. Suppose now that we have already found \( f_1, \ldots, f_k \), where \( k > 1 \), so that \( \{f_1, \ldots, f_k, g_{k+1}, \ldots, g_p\} = \text{id}(V) \) and \( W_j = \text{loc}(f_1, \ldots, f_j) \) is nonsingular if \( 1 < j < k \). Considering the linear system on \( W_k \) defined by
\[
\mu_1 g_{k+1} + \sum_{k < i < p} \mu_{ij} z_j^{d_i - d_i} g_i
\]
we deduce, again from the parametric transversality theorem, that there are constants \( \mu_{ij} \) so that if
\[
f_{k+1} = g_{k+1} + \sum_{k < i < p} \mu_{ij} z_j^{d_i - d_i} g_i
\]
then \( W_{k+1} = \text{loc}(f_1, \ldots, f_{k+1}) \) is nonsingular and obviously \( \{f_1, \ldots, f_{k+1}, g_{k+2}, \ldots, g_p\} = \text{id}(V) \).

Let \( H^d \) be a line bundle over \( \mathbb{P}_N(\mathbb{C}) \) defined by the linear system of hypersurfaces of degree \( d > 1 \) in \( \mathbb{P}_N(\mathbb{C}) \). It is well known that \( H^d \) has \( c_1(H^d) = dh \in H^2(\mathbb{P}_N(\mathbb{C}), \mathbb{Z}) \), where \( h \) is a generator of \( H^2(\mathbb{P}_N(\mathbb{C}), \mathbb{Z}) \).

Let \( V \) be a nonsingular complete intersection in \( \mathbb{P}_N(\mathbb{C}) \). We shall say that \( V \) has degrees \( (d_1, \ldots, d_p) \) if \( \text{id}(V) = \{f_1, \ldots, f_p\} \) and \( d_i = \deg f_i, 1 \leq i \leq p \). If \( j: V \to \mathbb{P}_N(\mathbb{C}) \) is the embedding of \( V \) in \( \mathbb{P}_N(\mathbb{C}) \), we shall write \( h \) for \( j^* h \).

Theorem 1. Let \( V \) be a nonsingular complete intersection in \( \mathbb{P}_N(\mathbb{C}) \) of degrees \( (d_1, \ldots, d_p) \), the total Chern class of \( V \) is given by
\[
c(V) = (1 + h)^{N+1}(1 + d_1 h)^{-1} \cdots (1 + d_p h)^{-1}.
\]

Proof. The proof follows from the previous lemma and from [11, 4.8.1] by induction, as in [11].

Remark. Applying the previous lemma, the other results of [10] or [11] about complete intersections of nonsingular hypersurfaces in general position are likewise valid for any nonsingular complete intersection.
3. The Euler characteristic. The following well-known lemma plays an important role in the development of this section.

**Lemma.** Let $V$ and $W$ be two $n$-dimensional nonsingular complete intersections in $\mathbb{P}_N(\mathbb{C})$ of the same degrees $(d_1, \ldots, d_p)$. Then $V$ and $W$ are diffeomorphic manifolds.

Since all the $n$-dimensional nonsingular complete intersections in $\mathbb{P}_N(\mathbb{C})$ of degrees $(d_1, \ldots, d_p)$ have, according to the above lemma, the same Euler characteristic, we will write it as $\chi(d_1, \ldots, d_p; n)$; hence we can choose a particularly simple complete intersection, so that:

**Lemma.** If $d_1, \ldots, d_p$ are integers $> 1$, for almost any $(a_j)_j$, $1 < j < p$, $j - 1 < i < N$, $a_j \in \mathbb{C}$, there is a nonsingular complete intersection $V$ in $\mathbb{P}_N(\mathbb{C})$ of degrees $(d_1, \ldots, d_p)$, with ideal

$$\text{id}(V) = \left\{ \sum_{i=j-1}^{N} a_j z_i^p; 1 < j < p \right\}$$

and such that $V \cap \{z_0 = 0\}$ is nonsingular.

**Proof.** It is again a simple application of the parametric transversality theorem, since, if we consider the mappings

$$F_1: \mathbb{C}^p \to C^{\infty}(\mathbb{C}^{N+1} - \{0\}, \mathbb{C}^p),$$

$$(a_j) \to f(z_0, \ldots, z_N) = \left( \sum_{i=0}^{N} a_1 z_i^{d_1}, \ldots, \sum_{i=p-1}^{N} a_p z_i^{d_p} \right),$$

and

$$F_2: \mathbb{C}^p \to C^{\infty}(\mathbb{C}^N - \{0\}, \mathbb{C}^p),$$

$$(a_j) \to g(z_1, \ldots, z_N) = \left( \sum_{i=1}^{N} a_1 z_i^{d_1}, \ldots, \sum_{i=p-1}^{N} a_p z_i^{d_p} \right),$$

we immediately verify that $\emptyset(F_1, \{0\})$ and $\emptyset(F_2, \{0\})$ are open dense sets so that their intersection will also be dense.

**Theorem 2.** With the same notations as above,

$$\chi(d_1, \ldots, d_p; n) = d_1 d_2 \cdots d_p \left\{ \sum_{i=0}^{n} (-1)^{n-i} \binom{N + 1}{i} A_p^{n-i} \right\}$$

where

$$A_p^k = \sum_{|a|=k} d_1^{a_1} d_2^{a_2} \cdots d_p^{a_p}.$$ 

**Proof.** Let $W = \{z \in \mathbb{P}_{N-1}^*; \sum_{i=j-1}^{N} a_i z_i^p = 2 < j < p\}$, $V_0 = V \cap \{z_0 = 0\}$ and $V_a = V - V_0$. The map $f: V_a \to W - V_0$ defined by

$$f([z_0, \ldots, z_N]) = [z_1, \ldots, z_N]$$

is a $d_i$-sheeted covering projection, therefore $\chi(V_a) = d_i \chi(W - V_0)$.
according to the Lefschetz duality theorem, we have
\[ \chi(W - V_0) = \chi(W, V_0) = \chi(W) - \chi(V_0) \]
and also
\[ \chi(V) = \chi(V_0) + \chi(V_0); \]
therefore
\[ \chi(V) = d_1 \chi(W) - (d_1 - 1)\chi(V_0). \]
We thus obtain the induction formula
\[ \chi(d_1, \ldots, d_p; n) = d_1 \chi(d_2, \ldots, d_p; n) - (d_1 - 1)\chi(d_1, \ldots, d_p; n - 1) \]
from which the theorem follows easily, starting from the initial values
\[ \chi(P_n(C)) = n + 1 \text{ and } \chi(d_1, \ldots, d_p; 0) = d_1 d_2 \cdots d_p. \]

**Remark.** A noteworthy consequence of the previous theorem is that the results contained in [2], [3], [4] and [5] are also valid for any nonsingular complete intersection.

4. Some applications. We can now apply the anterior result in order to obtain:

**Theorem 3** (Compare with [14]). If \( V \) is an \( n \)-dimensional nonsingular complete intersection in \( \mathbb{P}_n(C) \) of degrees \( (d_1, \ldots, d_p) \), its class is given by
\[ p_n(d_1, \ldots, d_p; n) = d_1 d_2 \cdots d_p \left\{ \sum_{i=0}^{n} (-1)^i \binom{N-1}{i} A_p^{n-i} \right\}. \]

**Proof.** It is known (see [12]) that the dual variety \( V^* \) of a nonsingular complete intersection \( V \) in \( \mathbb{P}_n(C) \) is a hypersurface in the dual projective space \( \mathbb{P}_N^n(C) \) of the hyperplanes; therefore the class of \( V \), i.e. the degree of \( V^* \), will be obtained as the intersection number of \( V^* \) with a general line of \( \mathbb{P}_N^n(C) \). But if we interpret this line as a pencil of hyperplanes in \( \mathbb{P}_N^n(C) \) and apply the Zeuthen-Segre formula in [1], we have
\[ \chi(V) = 2\chi(V \cap H_0) - \chi(V \cap H_0 \cap H_1) + (-1)^n p_n(V) \]
where \( H_0 \) and \( H_1 \) are two generic hyperplanes of the pencil, from which it follows:
\[ p_n(V) = (-1)^n \left\{ \chi(d_1, \ldots, d_p; n) - 2\chi(d_1, \ldots, d_p; n - 1) \right. \]
\[ \left. + \chi(d_1, \ldots, d_p; n - 2) \right\} \]
and applying Theorem 2, we obtain the expression for which we were aiming.

Finally, we apply Theorem 2 to obtain the Milnor number of the germs of homogeneous complete intersection with an isolated singularity.

Let \((X, 0)\) be an \( n \)-dimensional germ of complete intersection in \( \mathbb{C}^N \), defined in a neighborhood \( U \) of \( 0 \) by the functions \( f_1, \ldots, f_p \); we shall say that \((X, 0)\) is an homogeneous complete intersection of degrees \( (d_1, \ldots, d_p) \) if \( f_i \) is an homogeneous polynomial of degree \( d_i \), \( 1 < i < p \). From this it follows immediately that in this case we can take all \( \mathbb{C}^N \) as \( U \). We assume now that
(X, 0) has an isolated singularity, and we denote by \( f: \mathbb{C}^N \rightarrow \mathbb{C}^p \) the mapping with components \( f_1, \ldots, f_p \); the affine manifold \( f^{-1}(y) \), where \( y \) is a regular value of \( f \), is called the Milnor fibre of \((X, 0)\) (see [13] and [8]). From [8] we thus know that \( f^{-1}(y) \) has the homotopy type of a bouquet of \( n \)-dimensional spheres; the number of spheres in this bouquet is called the Milnor number of \((X, 0), \mu(X, 0)\). The following result is originally stated in [7] (see also [6]).

**Theorem 4.** If \((X, 0)\) is an \( n \)-dimensional homogeneous complete intersection in \( \mathbb{C}^N \), of degrees \((d_1, \ldots, d_p)\) with an isolated singularity, we have

\[
\mu(X, 0) = (-1)^{n+1} + d_1 d_2 \cdots d_p \left\{ \sum_{i=0}^{n} (-1)^i \binom{N}{i} A_p^{n-i} \right\}.
\]

**Proof.** Let \( F = f^{-1}(y) \) be the Milnor fibre. Using a change of coordinates in \( \mathbb{C}^p \), if necessary, we may assume that \( y = (1, 1, \ldots, 1) \). Since \( f \) is homogeneous, we may define the projective variety \( V \) of \( \mathbb{P}_N(\mathbb{C}) \) as

\[
V = \{ z \in \mathbb{P}_N(\mathbb{C}); f_1(z_1, \ldots, z_N) - z_0^d = 0, 1 < i < p \}
\]

so \( V \) is a nonsingular complete intersection of degrees \((d_1, \ldots, d_p)\) and, if \( V_0 = V \cap \{ z_0 = 0 \} \), we shall get \( F = V - V_0 \). But since \( V_0 \) is a complete nonsingular intersection in \( \mathbb{P}_{N-1}(\mathbb{C}) \), of degrees \((d_1, \ldots, d_p)\), we shall have

\[
\chi(F) = \chi(V, V_0) = \chi(V) - \chi(V_0)
\]

i.e.

\[
1 + (-1)^n \mu = \chi(F) = \chi(d_1, \ldots, d_p; n) - \chi(d_1, \ldots, d_p; n-1)
\]

Now applying Theorem 2, we have the formula we sought to obtain.

**Bibliography**


*Departamento de Matemáticas, ETSII, Universidad Politécnica de Barcelona, Diagonal, 647, Barcelona (14), Spain*