SHORTER NOTES

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A PROOF OF THE PRINCIPLE OF LOCAL REFLEXIVITY

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Abstract. A quite elementary proof of the principle of local reflexivity is given.

Our purpose here is to give a proof of the "principle of local reflexivity" (using only the various forms of the Hahn-Banach theorem) as given in [1] and in an improved version in [2]. Our notation is standard. By $X$, $Y$ and $Z$ we shall denote Banach spaces and $J_X$ will denote the canonical embedding of $X$ into its second dual $X''$. An operator is a continuous linear function.

We shall require only the three following lemmas.

Lemma 1. Let $T: X \rightarrow Y$ be a closed operator. If $x''$ is in $X''$ and $y$ is in $Y$ such that $T''x'' = J_Yy$ then, for any $\delta > 0$ there exists an $x$ in $X$ such that

$$\|x\| < (1 + \delta)\|x''\|$$

and $Tx = y$.

Lemma 2. Let $T: X \rightarrow Y$ and $S: X \rightarrow Z$ be operators such that $T$ is closed and $S$ has finite rank. Then $U: X \rightarrow Y \times Z$ defined by $Ux = (Tx, Sx)$ is a closed operator.

Lemma 3. Let $0 < \delta < \frac{1}{4}$ and $T: X \rightarrow Y$ be an operator such that $X$ is finite dimensional and

$$(1 + \delta)^{-1} < \|Tx_i\| < (1 + \delta)$$

where $\{x_i\}$ is any $\delta$-net for the unit sphere of $X$. Then $T$ is invertible and

$$\|T\| \|T^{-1}\| < \left( \frac{1 + \delta}{1 - \delta} \right) \left( \frac{1}{1 + \delta} - \frac{\delta(1 + \delta)}{1 - \delta} \right)^{-1} = \vartheta(\delta).$$

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Lemma 1 follows immediately from the separation theorem. Lemma 2 is easily proved by observing that $U'$ is closed and Lemma 3 is a routine computation using the triangle inequality.

**Theorem [1], [2].** Let $E$ and $F$ be finite dimensional subspaces of $X''$ and $X'$, respectively, and let $\epsilon > 0$. Then there exist an operator $T: E \to X$ such that
\[
\|T\| \cdot \|T^{-1}\| < 1 + \epsilon, x'(Tx'') = x''(x') \text{ for all } x'' \text{ in } E \text{ and all } x' \text{ in } F, \text{ and } Tx'' = x \text{ if } J_{X'}X = x'' \text{ is in } E.
\]

**Proof.** Choose $\vartheta > 0$ so that $\vartheta(\vartheta) < 1 + \epsilon$ where $\vartheta$ is as in Lemma 3. Choose one element $a'_1, a'_2, \ldots, a'_m$ in $X'$ containing a basis of $F$ and such that
\[
\|x''\| < (1 + \vartheta) \sup_j |x''(a'_j)|
\]
for all $x''$ in $E$. Choose $b''_1, b''_2, \ldots, b''_n$ a $\vartheta$-net for the unit sphere of $E$ such that $b''_1, \ldots, b''_n$ is a basis for $J_{X'}X \cap E$ and $b''_1, \ldots, b''_r, r > k$, is a basis for $E$. Then, for $1 < p < q = n - r$, we have the unique scalars $\{t_{p,j}\}, 1 < i < r$, such that
\[
b''_r + p = \sum_{1 < i < r} t_{p,i} b''_i.
\]
Define for $1 < p < q$
\[
s_{p,i} = \begin{cases} 
 t_{p,i}, & i < r, \\
 -1, & i = r + p, \\
 0, & r < i < n \text{ and } i \neq r + p.
\end{cases}
\]
Define the operator $A_0: X'' \to X^{k+q}$ by
\[
A_0(x_1, \ldots, x_n) = (x_1, \ldots, x_k; \left( \sum_{1 < i < n} s_{p,i} x_i \right))
\]
for $1 < p < q$ where $X''$ and $X^{k+q}$ are the usual product spaces with the sup norm. The operator $A_0$ is onto since the matrix $(s_{p,i})$ has rank $q$. Define $A: X^n \to Z = X^{k+q} \times C^m$ by
\[
A(x_1, \ldots, x_n) = (A_0(x_1, \ldots, x_n); (a'_j(x_i)))
\]
for $1 < j < m$ and $1 < i < n$. By Lemma 2, $A$ is a closed operator. Observe that $A''(b''_1, \ldots, b''_n)$ is in $J_ZZ$. Therefore, by Lemma 1, there exists $(b_1, \ldots, b_n)$ in $X^n$, such that $J_ZA(b_1, \ldots, b_n) = A''(b''_1, \ldots, b''_n)$. Define the operator $T: E \to X$ such that $Tb_i'' = b_i$ for $1 < i < r$. For $1 < p < q$, we have that $\sum_{1 < i < n} s_{p,i} b_i'' = 0$ and $\sum_{1 < i < n} s_{p,i} b_i = 0$ which gives that $Tb_i'' = b_i$ also for $r < i < n$. To apply Lemma 3 and complete the proof we need only observe that for each $i$,
\[
\|Tb_i''\| > \sup_j |a'_j(Tb_i'')| = \sup_j |b''_i(a'_j)| > (1 + \vartheta)^{-1}.
\]
This proof was presented at the Functional Analysis Conference at Oberwolfach in October, 1974.

REFERENCES


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