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INCIDENCE RINGS WITH SELF-DUALITY

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ABSTRACT. An artinian ring R is said to have self-duality if there is a Morita duality between the categories of left and right finitely generated R-modules. Here it is shown that the incidence ring of a finite preordered set over a division ring has self-duality. This is accomplished in part by calculating their injective modules.

Theorems of Azumaya [3], Morita [**11**], and Tachikawa [**13**] in the late 1950's give necessary and sufficient conditions to insure that the category of finitely generated left modules over a (necessarily left artinian) ring R is dual to the category of finitely generated right modules over a ring S. An open question in the subject of Morita duality for artinian rings is that of characterizing those artinian rings, other than artin algebras and QF rings, that have *self-duality*, that is, those for which there exists a duality between their categories of left and right finitely generated modules. Recently, artinian rings with self-duality have been shown to include certain factors of skew polynomial rings [**12**], hereditary artinian tensor rings satisfying the Dlab-Ringel duality conditions [2], rings with quivers that are trees [**8**] and many serial rings [9]. Here we prove that the incidence ring of a finite preordered set over a division ring (hereafter called a *finite incidence ring*) has self-duality. This class of rings properly contains the class of hereditary serial rings and, indeed, all artinian rings with quivers that are trees [**8**]. The self-duality constructed is weakly symmetric, so we may apply [**9**, (4.1)] to show that any factor ring of a finite incidence ring has a (weakly symmetric) self-duality.

A finite incidence ring over a division ring D can be characterized as a (unital) subring of the $(n \times n)$-matrix ring over D satisfying $R = \sum\{DI_{kl} | I_{kk}RI_{ll} \neq 0\}$, where I_{kl} is the matrix unit with 1 in the (k, l)-position and 0 elsewhere. (Thus, finite incidence rings coincide with Mitchell's tic tac toe rings over division rings [**10**].) To prove that an artinian ring R has self-duality, it is sufficient to show that the basic ring eRe of R is isomorphic to the endomorphism ring of the minimal left injective cogenerator over eRe [3], [**11**]. It is not hard to show that the basic ring of a finite incidence ring is a finite incidence ring over the same division ring. Henceforth, we let R be a basic indecomposable $(n \times n)$-finite incidence ring over the division ring D. We shall consider D as the subring of constant diagonal matrices in R. Let $e_k = I_{kk}$ be the matrix unit with 1 in the (k, k)-position and 0 elsewhere. Then $e_k \in R$. If $I_{kl} \in R$, let $e_{kl} = I_{kl}$; if not, let $e_{kl} = 0$. The radical of R is

$$J = J(R) = \sum\{De_{ij} | i \neq j\}.$$

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We denote the composition length of a module N by $c(N)$ and its injective envelope by $E(N)$. Let M be the set of $(n \times 1)$-column vectors over D, with a left R-structure given by considering R as a subring of the $(n \times n)$-matrix ring over D. Let $m_k = (\delta_{k,i})_i \in M$. Define a monomorphism ι_k: $Re_k \to M$ by ι_k: $re_k \mapsto re_k m_k = rm_k$. Since $c({}_{e_kRe_k}e_kM) = 1$, we conclude from [4] that M and hence also Re_k are distributive (have a distributive lattice of submodules) for all k. Similarly, e_kR is distributive for each k.

Now consider any submodule N of M, and let $x \in N$ with $x = \sum_{k=1}^n d_k m_k$. If $d_k \neq 0$, then $e_k d_k^{-1} x = e_k d_k^{-1} d_k m_k = m_k$, so $m_k \in N$. Thus $x = \sum_{k=1}^n d_k m_k = \sum_{m_k \in N} d_k m_k$, so

$$N = \sum \{ Dm_k | m_k \in N \}.$$

Define submodules $L_k \subset M$ by $L_k = \Sigma\{Rm_j | e_kRe_j = 0\}$. If $e_kRe_j \neq 0$, then $e_{kj} \neq 0$; thus $e_{kj} = 0$ implies $m_j \in L_k$. Conversely, suppose that $m_j \in L_k$ and write $m_j = \Sigma r_i m_i$ with $e_{ki} = 0$ and $r_i \in R$. Then $e_j m_j = \Sigma e_j r_i m_i$, so there exists i with $e_j r_i e_i m_i \neq 0$ and $e_{ki} = 0$. Thus $e_j r_i e_i \neq 0$, so $e_{ji} \neq 0$. But now if $e_{kj} \neq 0$, then $0 \neq e_{kj} e_{ji}$ and $e_{ki} \neq 0$, a contradiction. Hence $m_j \in L_k$ implies $e_{kj} = 0$, and

$$L_k = \sum \{ Dm_j | e_{kj} = 0 \}.$$

Let $E_k = M/L_k$ and let η_k be the natural epimorphism.

1. Proposition. *The module $E_k = M/L_k$ is the injective envelope of Re_k/Je_k.*

Proof. We first show that $\mathrm{Soc}(E_k) \cong Re_k/Je_k$. Let $x = \Sigma d_j m_j + L_k$ be any nonzero element of E_k and suppose that $d_i \neq 0$ with $i \neq k$ and $e_{ki} \neq 0$. Then $0 \neq d_i m_k + L_k = e_{ki} \Sigma d_j m_j + L_k \in Jx$. Therefore $x \notin \mathrm{Soc}(E_k)$. So $e_i\mathrm{Soc}(E_k) = 0$ if $e_{ki} \neq 0$ and $k \neq i$, and of course $e_i\mathrm{Soc}(E_k) = 0$ if $e_{ki} = 0$ (so that $m_i \in L_k$). Since $\mathrm{Soc}(E_k) \neq 0$, we must have $e_k\mathrm{Soc}(E_k) \neq 0$. Since M is distributive, so also is E_k; thus $\mathrm{Soc}(E_k) \cong Re_k/Je_k$. To conclude we may apply [7, Lemma 5] and [6, Lemma 2.3] to see that the lattice of submodules of $E(Re_k/Je_k)$ is isomorphic to that of e_kR_R. Hence

$$\begin{aligned} c(E(Re_k/Je_k)) = c(e_kR_R) &= \sum_{j=1}^n c\left(e_kRe_{je_jRe_j}\right) \\ &= \#\{j | e_kRe_j \neq 0\} = n - \#\{j | e_kRe_j = 0\} = c(E_k). \end{aligned}$$

Because $\mathrm{Soc}(E_k) \cong Re_k/Je_k$, E_k is a submodule of $E(Re_k/Je_k)$ of the same length as $E(Re_k/Je_k)$, so $E_k = E(Re_k/Je_k)$.

Next we show that $\mathrm{End}(M)$ is a division ring.

2. Proposition. *Let* g: $M \to M$.

(1) *If* $\ker g \neq 0$ *then* $g = 0$.

(2) *If* $\operatorname{im} g \neq M$ *then* $g = 0$.

Thus $\mathrm{End}(M)$ *is a division ring. Moreover*, $\mathrm{End}(M) \cong D$.

PROOF. If $\ker g \neq 0$, choose e_{j_0} so that $e_{j_0} \ker g \neq 0$, i.e., so that $m_{j_0} \in \ker g$. Let m_h be given. We will show that $m_h g = 0$. Since R is indecomposable, there exist $e_{i_1}, \ldots, e_{i_l}$ and $e_{j_1}, \ldots, e_{j_l} = e_h$ such that $e_{j_{k-1}} R e_{i_k} \neq 0 \neq e_{j_k} R e_{i_k}$ for $k = 1, \ldots, l$. (See [**1**, §7].) Suppose that $m_{j_{k-1}} g = 0$. Then also $m_{i_k} g = 0$, for if $m_{i_k} g \neq 0$, then $0 \neq e_{j_{k-1} i_k} e_{i_k}(m_{i_k} g) = m_{j_{k-1}} g$. And now $m_{j_k} g = (e_{j_k i_k} m_{i_k}) g = e_{j_k i_k}(m_{i_k} g) = 0$. Thus by induction $m_h g = 0$ for all h and $Mg = 0$.

For (2), note that if $\operatorname{im} g \neq M$, then $\ker g \neq 0$, so by (1), $g = 0$.

For the moreover part, let $f\colon M \to M$ and let $S = Rm_k$ be a simple submodule of M. Since M is distributive, $\operatorname{Soc}(M)$ is square-free so $Sf \subset S$. Thus we may regard $f|_S\colon S \to S$ as an $e_k R e_k$-map. But then for some $d \in D$, $d_k m_k f|_S = d_k d m_k$ for all $d_k m_k \in S$. Let $f'\colon M \to M$ via $f'\colon m \mapsto md$. Then $0 \neq S \subset \ker(f' - f)$, so by (1), $f' = f$ and the ring monomorphism $\Phi\colon D \to \operatorname{End}(M)$ via $\Phi(d)\colon m \mapsto md$ is onto. Hence $D \cong \operatorname{End}(M)$.

In our proof that finite incidence rings have self-duality, the main technique is that of changing the range or domain of a function. For example, if N is a distributive artinian module, L is a submodule of N and f is a map $f\colon L \to N$, then $\operatorname{im} f \subset L$, so we may regard f as a map from L to L. Dually, if N is a distributive noetherian module, L is a submodule of N and f is any map $f\colon N \to N/L$, then $\ker f \supset L$, so we may regard f as a map from N/L to N/L by the factor theorem. (See [**5**, §4.1].) These results allow us to develop the principal tool used in the proof of Theorem 4.

3. LEMMA. (1) *Let L be a nonzero indecomposable submodule of M and let f: $L \to M$. Then there exists a unique map $f'\colon M \to M$ such that $lf' = lf$ for all $l \in L$.*

(2) *Let $L = M/K$ be a nonzero indecomposable factor of M and let $f\colon M \to L$. Then there exists a unique map $f'\colon M \to M$ such that $mf' + K = mf$ for all $m \in M$.*

PROOF. (1) Let L be a nonzero indecomposable submodule of M and let $e = \Sigma\{e_j | e_j L \neq 0\}$. Since $c({}_{e_j R e_j} e_j M) = 1$ for all j, either $e_j L = 0$ or $e_j L = e_j M$. Thus $L = eM$. Let $f\colon L \to M$. By the remarks preceding the lemma, $Lf \subset L$. Let f^* denote f with range restricted to L. Now eRe is a finite incidence ring over D, $L = eM$ plays the role of M for eRe and $f^* \in \operatorname{End}({}_{eRe} eM)$. Since eM is indecomposable over R, it is an indecomposable module over eRe. Since eM is also faithful over eRe, eRe is an indecomposable ring and we may apply Proposition 2 to see that f^* is right multiplication by some $d \in D$. Define $f'\colon M \to M$ via $f'\colon m \mapsto md$. Then f' extends f. If $g\colon M \to M$ also extends f, then $0 \neq L \subset \operatorname{Ker}(g - f')$, so by Proposition 2, $g = f'$ and f' is unique.

(2) Let $L = M/K$ be a nonzero indecomposable submodule of M, let $e = \Sigma\{e_j | e_j L \neq 0\}$ and let $f\colon M \to L$. By the remarks preceding the lemma, $K \subset \ker f$, so we may define a map $f^*\colon L \to L$ via $f^*\colon m + K \mapsto mf$. Now eRe is a finite incidence ring over D, $L = e(M/K)$ plays the role of M for eRe and $f^* \in \operatorname{End}({}_{eRe} L)$. Since L is indecomposable over R, it is an indecomposable module over eRe. Since L is also faithful over eRe, eRe is indecomposable and we may apply Proposition 2 to see that for some $d \in D$, $(m + K)f^* = md + K$ for all $m \in M$. Define $f'\colon M \to M$ via $f'\colon m \mapsto md$. Then $mf' + K = md + K = (m + K)f^* = mf$

for all $m \in M$. If $g\colon M \to M$ also satisfies $mg + K = mf$ for all $m \in M$, then $m(g - f') \in K$ for all m, so $\operatorname{im}(g - f') \neq M$. Thus by Proposition 2, $g = f'$.

A duality D' between the categories of left and right finitely-generated R-modules is said to be *weakly symmetric* if for $J = \operatorname{rad}(R)$ and for e any primitive idempotent of R, $D'(Re/Je) \cong eR/eJ$. It is not hard to see that a ring isomorphism $\Phi\colon R \to \operatorname{End}({}_R E)$ such that $E\Phi(e) = E(Re/Je)$, for each primitive idempotent $e \in R$, induces such a weakly symmetric duality if R is artinian and ${}_R E$ is an injective cogenerator [**9**, (3.1)].

4. THEOREM. *Let R be a finite incidence ring over a division ring. Then R has a weakly symmetric duality.*

PROOF. We may assume that R is basic and indecomposable. Let the R-module M be defined as it has been throughout. Let $\iota_i\colon Re_i \to M$ be the natural monomorphism and $\eta_i\colon M \to M/L_i = E_i$ be the natural epimorphism as before. Let $E = \bigoplus_{i=1}^{n} E_i$ be the minimal injective cogenerator of R-mod and let $S = \operatorname{End}({}_R E)$ with $f_i \in S$ the natural projection onto E_i. Define $\theta\colon S \to R$ via $\theta\colon \Sigma f_i s f_j \mapsto \Sigma e_i r e_j$, where $e_i r e_j$ is defined below.

Consider the following commutative diagram:

$$\begin{tikzcd}
Re_i \arrow[r, "\iota_i"] \arrow[d, dotted, "e_i r e_j"'] & M \arrow[r, "\eta_i"] \arrow[d, dotted, "\delta"] & E_i \arrow[d, "f_i s f_j"] \\
Re_j \arrow[r, "\iota_j"] & M \arrow[r, "\eta_j"] & E_j
\end{tikzcd}$$

By Lemma 3 there exists a unique $\delta\colon M \to M$ such that $n_i f_i s f_j = \delta\eta_j$. If $f_i s f_j = 0$, then $\delta = 0$ and $e_i r e_j = 0$ is the only choice for $e_i r e_j$ to make the diagram commute. If $f_i s f_j \neq 0$, then $e_j E_i \neq 0$, so also $0 \neq e_i R e_j = \operatorname{Hom}(Re_i, Re_j)$ [**6**, Theorem 2.4]. Thus $e_{ij} \neq 0$, so $m_i = e_{ij} m_j$ and $\operatorname{im} \iota_i\delta \subset Rm_i \subset Rm_j$. Therefore the map $e_i r e_j$ exists uniquely with $e_i r e_j \iota_j = \iota_i \delta$. Thus, θ is a well-defined function. To see that θ is bijective, let $e_i r e_j \neq 0$ be given. By Lemma 3 there exists a unique $\delta\colon M \to M$ such that $e_i r e_j \iota_j = \iota_i \delta$. Now $e_i r e_j \neq 0$ implies that $e_{ij} \neq 0$, so that if $e_{jk} \neq 0$ then also $e_{ik} = e_{ij} e_{jk} \neq 0$. Thus $e_{ik} = 0$ implies $e_{jk} = 0$, so $L_i \subset L_j$. Since M is distributive, $L_i\delta \subset L_i \subset L_j \subset \operatorname{Ker} \delta\eta_j$. Hence $\delta\eta_j$ factors uniquely through η_i. Let $\eta_i f_i s i_j = \delta\eta_j$. Thus θ is bijective.

A simple argument shows that $\delta + \delta'$ and $e_i r e_j + e_i r' e_j$ are the maps associated with $f_i s f_j + f_i s' f_j$, and it follows that θ is additive. Since for each $i, j, k \in \{1, \cdots, n\}$, $f_i s f_j s' f_k$ corresponds to $\delta\delta'$ and then to $e_i r e_j r' e_k$, θ is multiplicative. Thus θ is a ring isomorphism. Also, f_i corresponds to 1_M corresponds to e_i, so $\theta(f_i) = e_i$, and R has a weakly symmetric duality.

If R is an artinian ring with a weakly symmetric duality and the primitive right (or left) ideals of R are distributive, then any factor ring of R has a weakly symmetric duality [**9**, (4.1)]. Since the primitive right (and left) ideals of a finite incidence ring are distributive, we may apply Theorem 4 to conclude that factor rings of finite incidence rings also have self-duality.

5. Corollary. *Any factor ring of a finite incidence ring over a division ring has a weakly symmetric duality.*

We note here that a finite incidence ring over a division ring is a tensor ring iff it is hereditary, and an artinian tensor ring is a finite incidence ring iff every principal right and left ideal generated by a primitive idempotent is distributive. Nonhereditary finite incidence rings are nontrivial factors of tensor rings with the same quivers, but since such tensor rings do not satisfy the hypotheses of [**9**, (4.1)], one cannot apply a result of Auslander, Platzeck, and Reiten [**2**], namely, that hereditary artinian tensor rings satisfying the Dlab-Ringel duality conditions have self-duality, to show directly that finite incidence rings have self-duality.

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