ON $p$-TORSION IN ETALE COHOMOLOGY 
AND IN THE BRAUER GROUP

ROBERT TREGER

Abstract. If $X$ is an affine scheme in characteristic $p > 0$, then $\text{Br}(X)(p) \cong H^2_{\text{et}}(X, G_m)(p)$ and $H^n_{\text{et}}(X, G_m)(p) = 0$ for $n > 3$. This gives a partial answer to the conjecture that the Brauer group of any scheme $X$ is canonically isomorphic to the torsion part of $H^2_{\text{et}}(X, G_m)$. This result is then applied to prove that $\text{Br}(R)(p)$ is $p$-divisible where $R$ is a commutative ring of characteristic $p > 0$ (theorem of Knus, Ojanguren and Saltman), and also to construct examples of domains $R$ of characteristic $p > 0$ with large $\text{Ker}(\text{Br}(R)(p) \to \text{Br}(Q)(p))$, where $Q$ is the ring of fractions of $R$.

The main result of this note (Theorem) is a partial answer to the following well-known Conjecture [4, II, 2]. The Brauer group of any scheme $X$ is canonically isomorphic to the torsion part of the second etale cohomology group of $X$ with coefficients in the sheaf of units $G_m$, i.e., the image of the inclusion $\delta: \text{Br}(X) \to H^2_{\text{et}}(X, G_m)[4, I, \text{Proposition 1.4}]$ coincides with the torsion part of $H^2_{\text{et}}(X, G_m)$.

The results of this note were obtained in Chicago in the fall of 1976 (cf. [9], [10]). O. Gaber, using a completely different approach, independently proved the following general result: the conjecture is true for $X = U_1 \cup U_2$ where $U_1, U_2$ are affine schemes. (I hope he will also publish his rather long but very interesting proof.)

1. Theorem. Let $X = \text{Spec}(R)$ be an affine scheme in characteristic $p > 0$. Then $\delta: \text{Br}(X)(p) \to H^2_{\text{et}}(X, G_m)(p)$ and $H^n_{\text{et}}(X, G_m)(p) = 0$ for $n > 3$.

Proof. For any positive integer $e$ we shall consider an extension of $R$ of the form $K_e = R[\{x_j | j \in J\}] / \{x_j^{p^e} - a_j | j \in J\}$ where $\{a_j | j \in J\}$ is a possibly infinite set of generators for the $R^{p^e}$-algebra $R$. Each algebra $K_e = \text{inj lim}_{\gamma \in I}(K_{\gamma,e})$, where

$K_{\gamma,e} = R[\{x_{\gamma,i} | i \in I\}] / \{x_{\gamma,i}^{p^e} - a_{\gamma,i} | 1 \leq i \leq s_{\gamma}\}$, 

and $\gamma \in I$, the index set. All $K_{\gamma,e}$ are free as $R$-modules.

Let $U$ be the functor which associates to any commutative $R$-algebra $S$ its group of units $U(S)$ [6, V, 1.2]. We denote by $H^n(S/R, U)$ or $H^n(S/R)$ the Amitsur

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The referee pointed out that Gaber's proof is a generalization of Hoobler's proof for smooth affine varieties over a field.

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cohomology groups for the functor $U[6, V, 1, 2]$. It follows immediately from the definition of the Amitsur complex and Berkson’s theorem (see a proof by D. Zelinski in [6, V, 5.1]) that

$$H^n(K_e/R) = H^n(K_{r,e}/R) = 0 \quad \text{for } n > 3, \quad \gamma \in \Gamma. \quad (1)$$

Let us consider a sequence of ring homomorphisms $R \xrightarrow{\epsilon} K_e \xrightarrow{p^e} R$, where $\epsilon$ is the natural embedding and $p^e$ is the $p^e$-power map. Let $Y_{r,e} = \text{Spec}(K_{r,e})$ and $Y_e = \text{Spec}(K_e)$. This sequence yields a sequence of homomorphisms of etale cohomology groups:

$$H^n_{\text{et}}(X, G_m) \xrightarrow{\epsilon} H^n_{\text{et}}(Y_e, G_m) \xrightarrow{p^e} H^n_{\text{et}}(X, G_m). \quad (2)$$

We shall need two general remarks. If $H^n_{\text{et}}(X, G_m)$ are the cohomology groups of $X$ in “fppf” topology [4, III, 5] then there exist canonical isomorphisms $H^n_{\text{et}}(X, G_m) \simeq H^n_{\text{fppf}}(X, G_m)$ [4, III, 11.7]. Moreover, $p^e \cdot \epsilon$ is the $p^e$-power map [6, V, 1].

Second, for any faithfully flat $K_e$-algebra $B$,

$$H^n(B/K_e) = H^n(K_e \otimes_R B/K_e)$$

[8, 4.3].

The classical construction of Rosenberg and Zelinsky shows that the natural map $H^2(K_e/R) \to H^2_{\text{et}}(X, G_m)$ factors through $\text{Br}(K_e/R)$ [6, V]. Consider the spectral sequence for the Amitsur complex [6, V, 4]. Since the homomorphism $\eta: \text{Ker} \to K_e$ given by $\eta(k_1 \otimes \ldots \otimes k_m) = k_1 \cdot \ldots \cdot k_m$ has a nilpotent kernel for each $m > 1$, by a theorem of Rosenberg and Zelinsky ([8, 4.1] or [6, V, 4]), the natural sequence

$$H^n(K_e, F_{\omega}/R) \to H^n(F_{\omega}/R) \to H^n(K_e \otimes_R F_{\omega}/K_e) \quad (3)$$

is exact for any etale $R$-algebra $F_{\omega} (\omega \in \Omega)$. Obviously, $K_e \otimes_R F_{\omega}$ are etale $K_e$-algebras. We want to compute $\lim_{\omega \in \Omega} H^n(K_e, F_{\omega}/R)$. Consider the second spectral sequence with $K = K_e, F = F_{\omega}$. By [8, Lemma 3.1], $E_1^{\cdot, n} = 0$ for all $n > 0, m = 0, 1$. Furthermore,

$$E_1^{3, n} = H^2(K_e \otimes_R F_{\omega}/F_{\omega})$$

and

$$E_2^{2, n} = H^1(K_e \otimes_R F_{\omega}/F_{\omega}) = \text{Pic}(K_e \otimes_R F_{\omega}/F_{\omega}).$$

Artin’s theorem [1] implies that $\lim_{\omega \in \Omega} E_1^{m,n} = 0$ for $m > 2, n > 0$.

Passing to the limit over the directed family $F_{\omega} (\omega \in \Omega)$ in (3) and applying a standard result about spectral sequences [2, XV, Theorem 5.12], we get the exact sequence

$$H^n(K_e/R) \xrightarrow{\alpha} H^n_{\text{et}}(X, G_m) \xrightarrow{\epsilon} H^n_{\text{et}}(Y_e, G_m). \quad (4)$$

Let $\xi \in \rho H^2_{\text{et}}(X, G_m)$, i.e. $\xi$ has order $p^e$ in the group $H^2_{\text{et}}(X, G_m)$. Then $p^e \cdot \epsilon(\xi) = 0$ (see (2)) hence, by a well-known lemma [4, III, 11.8], $\epsilon(\xi) = 0$. Therefore, $\xi \in \text{Im}(\alpha)$. Hence, $\xi$ comes from an Azumaya $R$-algebra.

If $n > 3$ then, by (1), $H^n(K_e/R) = 0$. Hence $\xi \in \rho H^2_{\text{et}}(X, G_m)$ implies $\xi = 0$. This proves the theorem.
2. We now give an example of a local ring $R$ of an affine domain over an algebraically closed field with large group $\text{Br}(Q/R)(p) \overset{\text{def}}{=} \text{Ker}(\text{Br}(R)(p) \to \text{Br}(Q)(p))$, where $Q$ is the quotient field of $R$.

Let $k$ be an algebraically closed field of characteristic $p > 0$. Let $(R, m)$ be a two-dimensional local normal $k$-domain with the quotient field $Q$ and residue field $R/m \simeq k$. There is a commutative diagram $[4, II, 1.7$ and $(7 \text{ bis})]$:

$\begin{array}{ccc}
0 & \to & \text{Br}(Q/R) \to \text{Br}(R) \to \text{Br}(Q) \\
\tau \downarrow & & \delta \downarrow \\
0 & \to & \text{Cl}(R^h)/\text{Cl}(R) \to H^2_{et}(X, \mathbb{G}_m) \to \text{Br}(Q)
\end{array}$

where $\tau$ is the restriction of $\delta$ and $R^h$ is the henselisation of $R$. It is well known that $\text{Cl}(R^h) = \text{Cl}(\hat{R})$, where $\hat{R}$ is the $m$-adic completion of $R$.

Suppose, now that $R$ (as above) is a factorial domain with a nonrational singularity. Then $\text{Cl}(R^h)/\text{Cl}(R) \simeq \text{Cl}(\hat{R})$, and $\text{Cl}(\hat{R})$ contains a nondiscrete commutative subgroup, hence $\text{Cl}(\hat{R})(p) \neq 0$. Thus $\text{Br}(Q/R)(p) \simeq \text{Cl}(\hat{R})(p) \neq 0$.

**Example.** Let $k$ (as above) be of characteristic 2. Let $A = k[X, Y, Z]/(X^2 + YZ^2 + XZ^2)$ where $(i, j) \neq (1, 1), (1, 2), (2, 1)$ and $(2i + 1, 2j + 1) = 1$. Let $R = A_m$ where $m = (x, y, z) \subset A$ is an ideal in $A$ generated by the images of $X, Y, Z$ in $A$. Then $R$ is a factorial domain but $\text{Cl}(R) = \mathbb{Z}$, where $\mathbb{Z}$ is asymptotic to $ij/2$, by Samuel [3, IV, 17]. Thus we can make $\text{Br}(Q/R)(p)$ as large as we wish.

Ojanguren (unpublished) independently constructed examples with nontrivial $\text{Br}(Q/R)$. M. Artin pointed out to me that one can construct examples in characteristic zero with nontrivial $\text{Br}(Q/R)$ by contracting some curves on algebraic $K - 3$ surfaces.

3. Now we present a short functorial proof of the following.

**Proposition (Knus-Ojanguren-Saltman; cf. [7]).** The Brauer group of any affine scheme $X$ in characteristic $p > 0$ is $p$-divisible.

I wish to thank D. Saltman for showing me his proof before it appeared in [7].

**Proof.** Let $X = \text{Spec}(R)$ and $K_1, Y_1 = \text{Spec}(K_1)$ be as in the theorem. There is a standard exact sequence of sheaves on $X_{f_1}$ (see, for instance, [5, 1.4])

$0 \to \mathbb{G}_{m,X} \overset{i}{\to} \mathbb{G}^1_{m, Y_1} \overset{\text{def}}{\to} Z^{1}_{Y_1/X} \overset{C - 1}{\to} \psi^*\mathbb{O}^{1}_{Y_1/X} \to 0$ (5)

where $\varphi: Y_1 \to X$ is the map defined by the inclusion: $R \to K_1$, $I$ is the formal $p$-power map, $\psi: X \to Y_1$ is the map induced by the map $p: K_1 \to R$, $C$ is the Cartier operator, and

$Z^{1}_{Y_1/X} = \text{Ker}\left[ \varphi_*d_{Y_1/X}: \varphi_*\mathbb{O}^1_{Y_1/X} \to \varphi_*\mathbb{O}^2_{Y_1/X} \right]$ is the sheaf of closed 1-forms. Consider the natural commutative diagram

$\begin{array}{ccc}
H^2_{f_1}(X, \mathbb{G}_{m,X}) & \overset{i}{\to} & H^2_{f_1}(X, \varphi_*\mathbb{G}_{m,Y_1}) \\
\downarrow F & & \downarrow W \\
H^2_{f_1}(X, \mathbb{G}_{m,X})
\end{array}$
where the map \( F \) is induced by the absolute Frobenius on \( X \) and \( W \) is induced by the map \( p \). Since \( X \) is affine and \( Z_{Y_{1/X}}^{1} \) and \( \psi_{*} \Omega_{Y_{1/X}}^{1} \) are quasi-coherent sheaves, \( H^{2}_{et}(X, \text{Ker}(C-I)) = 0 \). Hence \( i \) is surjective. It is trivial that \( W \) is surjective. Therefore \( F = W \cdot i \) is surjective. Since, by the theorem, \( \text{Br}(X)(p) \approx H^{2}_{et}(X, \mathbb{G}_{m}(p)) \), and \( H^{2}_{et}(X, \mathbb{G}_{m})(p) \approx H^{2}_{f}(X, \mathbb{G}_{m})(p) \) \( [4, \text{III}, 11.7] \), the Brauer group \( \text{Br}(X) \) is \( p \)-divisible.

Of course, for general schemes the Brauer group is not \( p \)-divisible (cf. \( [5, \S 2] \)).

4. Remark. Presumably our method can be applied to the investigation of \( p \)-torsion in the nonaffine cases (we used that \( X \) in the theorem is affine to conclude that \( \text{Ker}(\bar{p}^{*}) = 0 \) in (2)). A straightforward generalization of the theorem to curves can be used to prove an old theorem of M. Artin (unpublished): If \( f: V' \to V \) is a proper morphism with fibres of dimension 1 and \( V' \) regular of dimension 2, then \( R^{q}f_{*}\mathbb{G}_{m,V'} = 0 \) for \( q > 2 \). Indeed, the vanishing of \( (R^{q}f_{*}\mathbb{G}_{m,V'})(l) \), where \( l \) is any prime number, is proved exactly as the analogous result in \( [4, \text{III}, 3] \); see also \( [11] \). The theorem for curves takes care of the case \( l = p \), the characteristic of \( V \).

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Institute of Mathematics, Hebrew University, Jerusalem, Israel

Current address: School of Mathematics, Institute for Advanced Study, Princeton, New Jersey 08540