ON \( p \)-TORSION IN ÉTALE COHOMOLOGY
AND IN THE BRAUER GROUP

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Abstract. If \( X \) is an affine scheme in characteristic \( p > 0 \), then
\[ \text{Br}(X)(p) \simeq H^2_{\text{ét}}(X, \mathbb{G}_m)(p) \text{ and } H^n_{\text{ét}}(X, \mathbb{G}_m)(p) = 0 \text{ for } n > 3.\] This gives a partial answer to the conjecture that the Brauer group of any scheme \( X \) is canonically isomorphic to the torsion part of \( H^2_{\text{ét}}(X, \mathbb{G}_m) \). This result is then applied to prove that \( \text{Br}(R)(p) \) is \( p \)-divisible where \( R \) is a commutative ring of characteristic \( p > 0 \)
(theorem of Knus, Ojanguren and Saltman), and also to construct examples of domains \( R \) of characteristic \( p > 0 \) with large \( \text{Ker}(\text{Br}(R)(p) \to \text{Br}(Q)(p)) \), where \( Q \) is the ring of fractions of \( R \).

The main result of this note (Theorem) is a partial answer to the following well-known

Conjecture [4, II, 2]. The Brauer group of any scheme \( X \) is canonically isomorphic to the torsion part of the second étale cohomology group of \( X \) with coefficients in the sheaf of units \( \mathbb{G}_m \), i.e., the image of the inclusion \( \delta: \text{Br}(X) \to H^2_{\text{ét}}(X, \mathbb{G}_m) \) [4, I, Proposition 1.4] coincides with the torsion part of \( H^2_{\text{ét}}(X, \mathbb{G}_m) \).

The results of this note were obtained in Chicago in the fall of 1976 (cf. [9], [10]).

O. Gaber, using a completely different approach,\(^1\) independently proved the following general result: the conjecture is true for \( X = U_1 \cup U_2 \) where \( U_1, U_2 \) are affine schemes. (I hope he will also publish his rather long but very interesting proof.)

1. Theorem. Let \( X = \text{Spec}(R) \) be an affine scheme in characteristic \( p > 0 \). Then
\[ \delta: \text{Br}(X)(p) \to H^2_{\text{ét}}(X, \mathbb{G}_m)(p) \text{ and } H^n_{\text{ét}}(X, \mathbb{G}_m)(p) = 0 \text{ for } n > 3.\]

Proof. For any positive integer \( e \) we shall consider an extension of \( R \) of the form
\[ K_e = R[\{ x_j \mid j \in J \}]/(\{ x_j^{p^n} - a_j \mid j \in J \}) \]
where \( \{ a_j \mid j \in J \} \) is a possibly infinite set of generators for the \( R^{p^e} \)-algebra \( R \). Each algebra \( K_e = \text{inj lim}_{\gamma \in \Gamma}(K_{\gamma,e}) \), where
\[ K_{\gamma,e} = R[x_{\gamma,1}, \ldots, x_{\gamma,\delta}] / (\{ x_{\gamma,i}^{p^\alpha} - a_{\gamma,i} \mid 1 < i < s_{\gamma} \}), \quad a_{\gamma,i} \in \{ a_j \mid j \in J \} \]
and \( \gamma \in \Gamma \), the index set. All \( K_{\gamma,e} \) are free as \( R \)-modules.

Let \( U \) be the functor which associates to any commutative \( R \)-algebra \( S \) its group of units \( U(S) \) [6, V, 1.2]. We denote by \( H^n(S/R, U) \) or \( H^n(S/R) \) the Amitsur

\(^1\)The referee pointed out that Gaber’s proof is a generalization of Hoobler’s proof for smooth affine varieties over a field.
cohomology groups for the functor $U$ \cite{6, V, 1, 2}. It follows immediately from the definition of the Amitsur complex and Berkson’s theorem (see a proof by D. Zelinski in \cite{6, V, 5.1}) that

$$H^n(K_e/R) = H^n(K_{te}/R) = 0 \quad \text{for } n > 3, \quad \gamma \in \Gamma. \quad (1)$$

Let us consider a sequence of ring homomorphisms $R \xrightarrow{\varepsilon} K_e \xrightarrow{p^e} R$, where $\varepsilon$ is the natural embedding and $p^e$ is the $p^e$-power map. Let $Y_{te} = \text{Spec}(K_{te})$ and $Y_e = \text{Spec}(K_e)$. This sequence yields a sequence of homomorphisms of etale cohomology groups:

$$H^n_{\text{et}}(X, G_m) \xrightarrow{\varepsilon} H^n_{\text{et}}(Y_e, G_m) \xrightarrow{p^e} H^n_{\text{et}}(X, G_m). \quad (2)$$

We shall need two general remarks. If $H^n_{\text{et}}(X, G_m)$ are the cohomology groups of $X$ in “fppf” topology \cite{4, III, 5} then there exist canonical isomorphisms $H^n_{\text{et}}(X, G_m) \simeq H^{2n}(X, G_m) \cite{4, III, 11.7}$. Moreover, $\bar{p} \cdot \bar{\varepsilon}$ is the $p$-power map \cite{6, V, 1}.

Second, for any faithfully flat $K_e$-algebra $B$,

$$H^n(B/K_e) = H^n(K_e \otimes_R B / K_e) \quad [8, 4.3].$$

The classical construction of Rosenberg and Zelinsky shows that the natural map $H^2(K_e/R) \to H^2_{\text{et}}(X, G_m)$ factors through $\text{Br}(K_e/R) \ [6, V]$. Consider the spectral sequence for the Amitsur complex \cite{6, V, 4}. Since the homomorphism $\eta: K_e^m \to K_e$ given by $\eta(k_1 \otimes \cdots \otimes k_m) = k_1 \cdots k_m$ has a nilpotent kernel for each $m > 1$, by a theorem of Rosenberg and Zelinsky (\cite{8, 4.1} or \cite{6, V, 4}), the natural sequence

$$H^n(K_e, F_{\omega}/R) \to H^n(F_{\omega}/R) \to H^n(K_e \otimes_R F_{\omega}/K_e) \quad (3)$$

is exact for any etale $R$-algebra $F_{\omega}$ ($\omega \in \Omega$). Obviously, $K_e \otimes_R F_{\omega}$ are etale $K_e$-algebras. We want to compute $\lim_{\omega \in \Omega} H^n_{\text{et}}(K_e, F_{\omega}/R)$. Consider the second spectral sequence with $K = K_e$, $F = F_{\omega}$. By [8, Lemma 3.1], $"E_1"^{m,n} = 0$ for all $n > 0, m = 0, 1$. Furthermore,

$$"E_1^{3,n} = H^2(K_e \otimes_R F_{\omega}/F_{\omega})$$

and

$$"E_1^{2,n} = H^1(K_e \otimes_R F_{\omega}/F_{\omega}) \to \text{Pic}(K_e \otimes_R F_{\omega}/F_{\omega}).$$

Artin’s theorem \cite{1} implies that $\lim_{\omega \in \Omega} "E_1"^{m,n} = 0$ for $m > 2, n > 0$.

Passing to the limit over the directed family $F_{\omega}$ ($\omega \in \Omega$) in (3) and applying a standard result about spectral sequences \cite[XXV, Theorem 5.12]{2}, we get the exact sequence

$$H^n(K_e/R) \xrightarrow{\varepsilon} H^n_{\text{et}}(X, G_m) \xrightarrow{p^e} H^n_{\text{et}}(Y_e, G_m). \quad (4)$$

Let $\xi \in p^eH^2_{\text{et}}(X, G_m)$, i.e. $\xi$ has order $p^e$ in the group $H^2_{\text{et}}(X, G_m)$. Then $p^e \cdot \bar{\varepsilon}(\xi) = 0$ (see (2)) hence, by a well-known lemma \cite{4, III, 11.8}, $\bar{\varepsilon}(\xi) = 0$. Therefore, $\xi \in \text{Im}(\alpha)$. Hence, $\xi$ comes from an Azumaya $R$-algebra.

If $n > 3$ then, by (1), $H^n(K_e/R) = 0$. Hence $\xi \in p^eH^2_{\text{et}}(X, G_m)$ implies $\xi = 0$. This proves the theorem.
2. We now give an example of a local ring \( R \) of an affine domain over an algebraically closed field with large group \( \text{Br}(Q/R)(p) \overset{\text{def}}{=} \text{Ker}(\text{Br}(R)(p) \rightarrow \text{Br}(Q)(p)) \), where \( Q \) is the quotient field of \( R \).

Let \( k \) be an algebraically closed field of characteristic \( p > 0 \). Let \((R, m)\) be a two-dimensional local normal \( k\)-domain with the quotient field \( Q \) and residue field \( R/m \cong k \). There is a commutative diagram [4, II, 1.7 and (7 bis)]

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Br}(Q/R) & \rightarrow & \text{Br}(R) & \rightarrow & \text{Br}(Q) \\
\tau & \downarrow & \delta & \downarrow & \text{Cl}(R^h)/\text{Cl}(R) & \rightarrow & H^2_{et}(\mathbb{G}_m, \mathcal{O}_R) & \rightarrow & \text{Br}(R) \\
0 & \rightarrow & \text{Cl}(R^h)/\text{Cl}(R) & \rightarrow & H^2_{et}(X, \mathbb{G}_m) & \rightarrow & \text{Br}(R) \\
\end{array}
\]

where \( \tau \) is the restriction of \( \delta \) and \( R^h \) is the henselisation of \( R \). It is well known that \( \text{Cl}(R^h) = \text{Cl}(R) \), where \( \hat{R} \) is the \( m \)-adic completion of \( R \).

Example. Let \( k \) (as above) be of characteristic 2. Let \( A = k[[X, Y, Z]]/(X^2 + Y^2 + Z^2 + 1) \) where \( (i, j) \neq (1, 1), (1, 2), (2, 1) \) and \( (2i + 1, 2j + 1) = 1 \). Let \( R = A_m \) where \( m = (x, y, z) \subset A \) is an ideal in \( A \) generated by the images of \( X, Y, Z \) in \( A \). Then \( R \) is a factorial domain but \( \text{Cl}(R) = \text{Cl}(A^+) \) where \( A^+ \) is asymptotic to \( ij/2 \), by Samuel [3, IV, 17]. Thus we can make \( \text{Br}(Q/R)(p) \) as large as we wish.

Ojanguren (unpublished) independently constructed examples with nontrivial \( \text{Br}(Q/R) \). M. Artin pointed out to me that one can construct examples in characteristic zero with nontrivial \( \text{Br}(Q/R) \) by contracting some curves on algebraic \( K-3 \) surfaces.

3. Now we present a short functorial proof of the following.

**Proposition (Knus-Ojanguren-Saltman; cf. [7]).** The Brauer group of any affine scheme \( X \) in characteristic \( p > 0 \) is \( p \)-divisible.

I wish to thank D. Saltman for showing me his proof before it appeared in [7].

**Proof.** Let \( X = \text{Spec}(R) \) and \( K_1, Y_1 = \text{Spec}(K_1) \) be as in the theorem. There is a standard exact sequence of sheaves on \( X_f_1 \) (see, for instance, [5, 1.4])

\[
0 \rightarrow G_{m, X} \overset{i}{\rightarrow} \varphi^* G_{m, Y_1} \overset{d_{Y_1/X}}{\rightarrow} Z^1_{Y_1/X} \overset{\psi^* \Omega^1_{Y_1/X}}{\rightarrow} 0 \tag{5}
\]

where \( \varphi : Y_1 \rightarrow X \) is the map defined by the inclusion: \( R \rightarrow K_1 \), \( I \) is the formal \( p \)-power map, \( \psi : X \rightarrow Y_1 \) is the map induced by the map \( p : K_1 \rightarrow R \), \( C \) is the Cartier operator, and

\[
Z^1_{Y_1/X} = \text{Ker} \left[ \varphi^* d_{Y_1/X} : \varphi^* \Omega^1_{Y_1/X} \rightarrow \varphi^* \Omega^2_{Y_1/X} \right]
\]

is the sheaf of closed 1-forms. Consider the natural commutative diagram

\[
\begin{array}{cccc}
H^2_{f_1}(X, G_{m, X}) & \overset{i}{\rightarrow} & H^2_{f_1}(X, \varphi^* G_{m, Y_1}) & \overset{\psi}{\rightarrow} \\
H^2_{f_1}(X, G_{m, X}) & \overset{\psi}{\rightarrow} & \overset{w}{\rightarrow} & H^2_{f_1}(X, G_{m, X}) \\
\end{array}
\]
where the map $F$ is induced by the absolute Frobenius on $X$ and $W$ is induced by the map $p$. Since $X$ is affine and $Z_{Y_1/X}$ and $\psi_1^*\Omega_{Y_1/X}$ are quasi-coherent sheaves, $H^2_{et}(X, \ker(C - I)) = 0$. Hence $i$ is surjective. It is trivial that $W$ is surjective. Therefore $F = W \cdot i$ is surjective. Since, by the theorem, $\text{Br}(X)(p) \cong H^1_{et}(X, \mathbb{G}_m(p))$, and $H^2_{et}(X, \mathbb{G}_m)(p) \cong H^2_f(X, \mathbb{G}_m)(p)$ [4, III, 11.7], the Brauer group $\text{Br}(X)$ is $p$-divisible.

Of course, for general schemes the Brauer group is not $p$-divisible (cf. [5, §2]).

4. REMARK. Presumably our method can be applied to the investigation of $p$-torsion in the nonaffine cases (we used that $X$ in the theorem is affine to conclude that $\ker(p^*) = 0$ in (2)). A straightforward generalization of the theorem to curves can be used to prove an old theorem of M. Artin (unpublished): If $f: V' \to V$ is a proper morphism with fibres of dimension 1 and $V'$ regular of dimension 2, then $R^qf_*\mathbb{G}_{m,V'} = 0$ for $q > 2$. Indeed, the vanishing of $(R^qf_*\mathbb{G}_{m,V'})(l)$, where $l$ is any prime number, is proved exactly as the analogous result in [4, III, 3]; see also [11]. The theorem for curves takes care of the case $l = p$, the characteristic of $V$.

REFERENCES


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