HAUSDORFF MATRICES AS BOUNDED OPERATORS OVER $l$

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Abstract. A necessary and sufficient condition is obtained for an arbitrary Hausdorff matrix to belong to $B(l)$. It is then shown that every conservative quasi-Hausdorff matrix is of type $M$.

Let $(H, \mu)$ denote the Hausdorff method with generating sequence $\mu = \{\mu_n\}$, $l = \{(x_n) \mid \sum_n |x_n| < \infty\}$, $B(l)$ the algebra of bounded linear operators on $l$. A necessary and sufficient condition is obtained for an arbitrary Hausdorff method to belong to $B(l)$. It is then shown that every conservative quasi-Hausdorff matrix is of type $M$.

An infinite matrix is called triangular if it has only zeros above the main diagonal. Let $B(c)$ denote the algebra of bounded linear operators in $c$, the space of convergent sequences.

Lemma. Let $A$ be a triangular matrix satisfying:

1. $A \in B(l)$,
2. $t_0 = \sum_{k=0}^\infty |a_{nk}|$ is monotone increasing in $n$,
3. $\lim_n t_n$ exists, where $t_n = \sum_{k=0}^n a_{nk}$.

Then $A \in B(c)$.

Proof. Condition (1) is equivalent to $\sup_k \sum_{n=k}^\infty |a_{nk}| < \infty$, which implies $\sum_{n=k}^N |a_{nk}| < M$ for every $N > k$, where $M$ is independent of $N$ and $k$. Summing $k$ over $[0, N]$ yields $\sum_{k=0}^N \sum_{n=k}^N |a_{nk}| < M(N + 1)$. Interchanging the order of summation gives

$$ \sum_{n=0}^N t_n^* / (N + 1) < M. \quad (4) $$

The left-hand side of (4) is the $N$th term of the Cesàro transform of order one, $(C, 1)$, of the sequence $\{t_n^*\}$. Since $\{t_n^*\}$ is monotone increasing, the norm of $A$ in $B(c)$ is $\|A\|_c = \lim_n t_n^*$. If $\|A\|_c = \infty$, then the total regularity of $(C, 1)$ implies that the l.h.s. of (4) tends to $\infty$ as $N \to \infty$, a contradiction. Therefore $\|A\|_c$ is finite. Condition (1) implies $A$ has zero column limits and (3) assures the existence of the limit of the row sums. Therefore $A \in B(c)$.
Theorem 1. Let $H$ be a Hausdorff matrix. Then $H \in B(l)$ if and only if $\mu$ is a moment sequence satisfying
\[ \int_0^1 |d\beta(t)|/t < \infty. \quad (5) \]

Proof. In [3, p. 279] it is shown that, if $\mu$ is a moment sequence, then $H \in B(l)$ if and only if $\mu$ satisfies (5). The theorem in Hardy actually treats conservative quasi-Hausdorff methods $(H^*, \mu)$, but $(H^*, \mu)$ is merely the matrix transpose of $(H, \mu)$, so that the norm condition for the regularity of $H^*$ is the same as the norm condition for $H \in B(l)$.

It remains to show that, if $H \in B(l)$, then $\mu$ is a moment sequence.

From [3, p. 254] or [4, Lemma 2, p. 177], $\{t_*^n\}$ is monotone increasing in $n$. Each row sum of $H$ adds up to $\mu_0$, so (3) of the Lemma is satisfied. Applying the Lemma, $H \in B(c)$, which by [3, p. 260] is equivalent to $\mu$ being a moment sequence.

Let $A \in B(c)$, $A$ a matrix. Then $A$ is said to be of type $M$ if the only solution of $tA = 0$, $t \in l$, is $t = 0$.

Theorem 2. Let $(H^*, \mu) \in B(c)$. Then $(H^*, \mu)$ is of type $M$.

In [5] it was shown that $H^*$ is of type $M$, provided at most a finite number of the $\mu_n$ are zero. We now provide a different proof, which removes that restriction.

Since $H^*$ is the matrix transpose of $H$, the condition $H^* \in B(c)$ is equivalent to $H \in B(l)$. By Theorem 1, $\mu$ is a moment sequence. Thus there exists a function $\beta(t) \in BV[0, 1]$ such that $\mu_n = \int_0^1 t^n d\beta(t)$. Define
\[ F(z) = \int_0^1 t^z d\beta(t). \]
Then $F$ is analytic in $\text{Re } z > 0$ and continuous on $\text{Im } z = 0$. Moreover, if $b_i$ denote the real zeros of $F$ in $\text{Re } z > 0$, they satisfy $\Sigma 1/b_i < \infty$, by a result of Carleman [1]. To say that $H^*$ is of type $M$ is equivalent to saying that $H$ is 1-1 on $l$.

Since $H = \delta \mu \delta$, where $\delta$ is the triangular matrix of signed binomial coefficients and $\mu$ is a diagonal matrix with diagonal entries $\mu_n$, we may use associativity of multiplication and the fact that $\delta$ is its own inverse, to obtain $\mu \delta t = 0$; i.e., $\mu_n \Delta^n t_0 = 0$ for $n = 0, 1, 2, \ldots$. Now appeal to [2, Theorem 1] to conclude that $t$ is a constant sequence. The result follows, since the only constant sequence in $l$ is the zero sequence.

Theorems 1 and 2 are also true for generalized Hausdorff matrices of the form
\[ h_{nk} = \binom{n + \alpha}{n - k} \Delta^n - k \mu_k, \quad \text{with } \alpha > 0. \]
References

1. T. Carleman, Über die Approximation analytischer Funktionen durch Aggregate vorgegebener Potenzen, Ark. Mat. 17 (1922), no. 9.

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