A VOLTERRA EQUATION
WITH SQUARE INTEGRABLE SOLUTION

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Abstract. We study the asymptotic behavior of the solutions of the nonlinear Volterra integrodifferential equation

\[ x'(t) + \int_0^t a(t-s)g(x(s)) \, ds = f(t) \quad (t \in R^+). \]

Here \( R^+ = [0, \infty) \), \( a, g \) and \( f \) are given real functions, and \( x \) is the unknown solution. In particular, we give sufficient conditions which imply that \( x \) and \( x' \) are square integrable.

1. Introduction and summary of results. We study the asymptotic behavior of the solutions of the Volterra integrodifferential equation

\[ x'(t) + \int_0^t a(t-s)g(x(s)) \, ds = f(t) \quad (t \in R^+). \tag{1.1} \]

Here \( R^+ = [0, \infty) \), the prime denotes differentiation, \( a, g \) and \( f \) are given, real functions, and \( x \) is the unknown solution. In particular, we give sufficient conditions which imply that the solutions satisfy \( x, x' \in L^2(R^+) \).

Our assumptions are the following:

\[ a = b + c \text{ is strongly positive definite,} \tag{1.2} \]

where

\[ b \in L^1(R^+) \text{ satisfies } |\hat{b}(\omega)|^2 < \beta \Re \hat{b}(\omega) \quad (\omega \in R) \tag{1.3} \]

for some \( \beta > 0 \),

\[ c \text{ is positive definite, and } c' \in L^1 \cap BV(R^+), \tag{1.4} \]

\[ g \in C(R), \, \xi g(\xi) > 0 \quad (\xi \neq 0), \text{ and } \lim_{\xi \to 0} \inf g(\xi)/\xi > 0, \tag{1.5} \]

\[ f = f_1 + f_2 + f_3, \text{ where } f_1 \in L^2(R^+), f_2 \in BV(R^+), \tag{1.6} \]

and \( f_3 \in L^\infty(R^+), f'_3 \in L^2(R^+) \).

Here \( \hat{b}(\omega) = \int_0^\infty e^{-i\omega t} b(t) \, dt \) is the Fourier transform of \( b \). The strong positive definiteness of \( a \) means that there exists \( \epsilon > 0 \) such that the function \( a(t) - \epsilon e^{-t} \) is positive definite. The statements concerning \( c' \) and \( f'_3 \) should be interpreted as requirements that \( c, f_3 \) be locally absolutely continuous, together with the additional conditions on the derivatives. BV stands for functions of bounded variation.

We call \( x \) a solution of (1.1) if \( x \) is locally absolutely continuous, and (1.1) holds a.e. In addition to (1.2)–(1.6) above we shall have to assume that a solution \( x \) of
(1.1) satisfies

\[ x, Q_a \in L^\infty(R^+) \tag{1.7} \]

where

\[ Q_a(T) = \int_0^T g(x(t)) \int_0^t a(t-s)g(x(s)) \, ds \, dt \quad (T \in R^+) \tag{1.8} \]

Sufficient conditions for (1.7) to hold can be found in [7]. For example, any one of (1.9)–(1.11) below combined with (1.2)–(1.5) and the assumption

\[ - \int_{-\infty}^0 g(\xi) \, d\xi = \int_0^\infty g(\xi) \, d\xi = \infty \]

imply (1.7):

\[ f \in L^1(R^+), \text{ and } \limsup_{|\xi| \to \infty} |g(\xi)| \left(1 + \int_0^\xi g(\eta) \, d\eta\right)^{-1} < \infty \tag{1.9} \]

\[ f, f' \in L^2(R^+), \tag{1.10} \]

\[ c(\infty) > 0, \text{ and } f \in BV(R^+) \tag{1.11} \]

We prove the following result:

**Theorem 1.** Let (1.2)–(1.6) hold, and let \( x \) be a solution of (1.1) satisfying (1.7). Then \( x, x' \in L^2(R^+) \).

Theorem 1 is an improved version of [8, Theorem 1(iii)]. One gets [8, Theorem 1(iii)] by adding (1.10) and

\[ b \equiv 0, \quad c - c(\infty) \in L^1(R^+) \tag{1.12} \]

to the hypothesis of Theorem 1.

Theorem 1 extends some of the results in [5] and [6]. The hypothesis used here is comparatively strong, but, on the other hand, we now get the stronger conclusion \( x \in L^2(R^+) \) (which amounts to a faster convergence of \( x \) to zero than [5] and [6] yield).

This work may be regarded as a strengthening of [8], which in turn was inspired by some estimates in the two papers [1] and [2] of MacCamy. In spite of this fact our argument is quite different from MacCamy's. MacCamy does not work with a scalar equation as we do, but with an abstract Volterra equation of hyperbolic type. We shall return elsewhere [10] to the question of how the estimates in the proof of Theorem 1 should be modified in the abstract case.

We discuss conditions (1.2)–(1.4) in §3.

**2. Proof of Theorem 1.** Define

\[ \varphi(t) = g(x(t)), \quad d(t) = (1 + c(0))e^{-t} \quad (t \in R^+) \]

Let \( \ast \) denote convolution, subtract \( d \ast \varphi \) from both sides of (1.1), and use (1.2), (1.6) to get

\[ x' - (d - c) \ast \varphi - f_2 - f_3 = f_1 - (b + d) \ast \varphi \tag{2.1} \]

Define

\[ u = (d - c) \ast \varphi, \quad v = (d - c)' \ast \varphi, \quad w = (b + d) \ast \varphi \tag{2.2} \]
Multiply (2.1) by \( x' \), integrate over \((0, T)\), and integrate the terms on the left-hand side by parts (except the first one) to get

\[
\int_0^T \left[ x'(t) \right]^2 dt + \int_0^T x(t)g(x(t)) \, dt \\
= x(T)\left[ u(T) + f_2(T) + f_3(T) \right] - x(0)f_3(0) \\
- \int_0^T x(t) \, df_2(t) + \int_0^T x'(t)\left[ f_1(t) - w(t) \right] \, dt \\
- \int_0^T x(t)\left[ v(t) + f_1(t) \right] \, dt,
\]

where we have redefined \( f_2 \) so that it is continuous from the left, and \( f_2(0) = 0 \). By the H"older and Minkowski inequalities,

\[
\|x'\|_2^2 + \int_0^T x(t)g(x(t)) \, dt < (\|u\|_\infty + \|f_2\|_\infty + \|\int f_2(t)\|_1 + 2\|f_3\|_\infty)\|x\|_\infty \\
+ \left( \|f_1\|_2 + \|w\|_2 \right)\|x'\|_2 + (\|v\|_2 + \|f_1\|_2)\|x\|_2,
\]

(2.3)

where \( \| \cdot \|_p \) \((p = 2, \infty)\) is the norm of \( L^p(0, T)\), and \( \|f_2\|_1\) is the total variation of \( f_2 \).

We claim that

\[
u \in L^\infty(\mathbb{R}^+), \quad v, w \in L^2(\mathbb{R}^+).
\]

(2.4)

Assume this for the moment. Then, by (1.6), (1.7) and (2.4), inequality (2.3) is of the form

\[
\|x'\|_2^2 + \int_0^T x(t)g(x(t)) \, dt < C(1 + \|x'\|_2 + \|x\|_2),
\]

(2.5)

where \( C \) is a (sufficiently large) constant independent of \( T \). Observe that (1.5), (1.7) imply the existence of \( \epsilon > 0 \) such that \( x(t)g(x(t)) > \epsilon |x(t)|^2 \) \((t \in \mathbb{R}^+)\). Hence (2.5) becomes

\[
\|x'\|_2^2 + \epsilon \|x\|_2^2 < C(1 + \|x'\|_2 + \|x\|_2),
\]

from which the conclusion of Theorem 1 follows.

It remains to verify the crucial estimate (2.4). Observe that the functions \( b, c \) and \( d \) are all positive definite, and that by (1.2), \( 0 < Q_b(T) < Q_c(T) \), \( 0 < Q_c(T) < Q_d(T) \), and \( 0 < Q_d(T) < CQ_a(T) \), where \( Q_b, Q_c \) and \( Q_d \) are defined as in (1.8), and \( C \) is some positive constant. Thus (1.7) implies

\[
Q_b, Q_c, Q_d \in L^\infty(\mathbb{R}^+).
\]

(2.6)

Both \( c \) and \( d \) are continuous and positive definite, and so [4, Lemma 6.1] yields

\[
|c \cdot \varphi(T)|^2 < 2c(0)Q_c(T), \quad |d \cdot \varphi(T)|^2 < 2d(0)Q_d(T).
\]

(2.7)

Combining (2.2) with (2.6) and (2.7) one gets the first part of (2.4). By (1.3) and [5, Lemma 1],

\[
\|b \cdot \varphi\|_2^2 < \beta Q_b(T).
\]

(2.8)

Observe that \( c', d', d' \in L^1 \cap BV(\mathbb{R}^+) \), and use (1.2) and [9, Lemma 2.2] to get

\[
\|c' \cdot \varphi\|_2^2 + \|d \cdot \varphi\|_2^2 + \|d' \cdot \varphi\|_2^2 < CQ_a(T).
\]
for some constant $C$. Combining this with (1.7), (2.2), (2.6) and (2.8) we get the second part of (2.4). This completes the proof of Theorem 1.

3. Comments. The proof of Theorem 1 gives us, in fact, a little more than $x \in L^2(R^+)$, namely

$$\int_0^\infty x(t)g(x(t)) \, dt < \infty$$

(cf. (2.5)). If $\limsup_{\xi \to 0} g(\xi)/\xi < \infty$, then (3.1) is equivalent to $x \in L^2(R^+)$. However, if e.g., $g(\xi) = \xi^{1/3}$ (which satisfies (1.5)), then (3.1) becomes $x \in L^{4/3}(R^+)$. The conditions (1.2)–(1.4) require a splitting of $a$ into two parts, and given $a$ it is not always obvious how this splitting should be done. Some requirements are obvious: If $a$ or $a'$ is unbounded, then the unbounded part must go into $b$, and if $a$ is not integrable, then the nonintegrable part must go into $c$. Below we shall give some examples where the splitting succeeds. For example, in the following two cases (1.3) holds:

- $b \in L^1 \cap BV(R^+)$ is strongly positive definite,
- $b \in L^1(R^+)$, and $b, -b'$ are convex

(see [5, Theorem 2]). Thus, if, e.g., $a$ is strongly positive definite, and $a - a(\infty) \in L^1 \cap BV(R^+)$, then one can take $b = a - a(\infty), c = a(\infty)$ (the strong positive definiteness of $a$ implies the strong positive definiteness of $b$ in this case, and $a(\infty) > 0$). On the other hand, if $a' \in L^1 \cap BV(R^+)$, then one may choose $b \equiv 0, c = a$. An example where (3.3) is used is the following: Suppose that $a(t) = t^{-a}$ ($t \in R^+$), where $0 < a < 1$, and define $b(t) = t^{-a} - (1 + t)^{-a}, c(t) = (1 + t)^{-a}$ (cf. [3, Corollary 2.2]).

One way of simplifying the problem of how one should split $a$ into two parts is to modify (1.3), (1.4), and modify the proof of Theorem 1 accordingly. One can replace (1.3), (1.4) by

$$b \in L^1(R^+), and |\hat{b}(\omega)|^2 < \beta \Re \hat{a}(\omega) \ (\omega \in R)$$

for some $\beta > 0$.

$$c \in L^2(R^+), c' \in L^1 \cap BV(R^+).$$

Most of the proof of Theorem 1 remains valid. (2.6) should be replaced by

$$Q_e \in L^\infty(R^+),$$

where $e(t) = e^{-t}$ ($t \in R^+$), and (2.7), (2.8) by

$$|c * \varphi(T)|^2 < 2Q_e(T) \int_0^\infty (c^2(t) + |c'(t)|^2) \, dt,$$

$$|d * \varphi(T)|^2 < 2(1 + c(0))^2Q_e(T),$$

$$\|b * \varphi\|^2_{L^2} < \beta Q_e(T).$$

The proofs of (3.6), (3.7) are similar to the proofs of [4, Lemma 6.1] and [5, Lemma 1].

In (3.4), (3.5) it is no longer required that $b$ and $c$ be positive definite, which
clearly facilitates the splitting of $a$ into $b + c$. In particular, (3.2) can be weakened to $b \in L^1 \cap BV(R^+)$. On the other hand, the added condition $c \in L^2(R^+)$ prevents the use of (3.4), (3.5), e.g., when $a(t) = t^{-\alpha}$ with $0 < \alpha < \frac{1}{2}$. Also observe that (3.4), (3.5) exclude the possibility $a(\infty) > 0$.

REFERENCES


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