PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 78, Number 2, February 1980

DERIVATIVE MEASURES

CASPER GOFFMAN[1] AND FON-CHE LIU[2]

ABSTRACT. A characterization of those measures which are distribution derivatives is undertaken. For functions of n variables in BVC, the derivative measures are absolutely continuous with respect to Hausdorff $n-1$ measure. For functions in W_1^1 they are absolutely continuous with respect to n measure. For linearly continuous functions the derivative measures are zero for sets whose Hausdorff $n-1$ measure is finite. For $n = 1$, since $n - 1 = 0$, this reduces to the standard facts.

In one variable, the derivative measure μ of a function f of bounded variation can be any regular finite measure. If f is continuous then μ must be nonatomic. Finally, if f is absolutely continuous then μ is absolutely continuous with respect to Lebesgue measure.

In this note, we discuss the analogous situation for functions of several variables. The functions of bounded variation are replaced by those of bounded variation in the sense of Cesari. $f = f(x_1, \ldots, x_n)$ is of this type, designated BVC, if for each $i = 1, \ldots, n$ there is an f^i, equivalent to f, which is of bounded variation in x_i for almost all values of the remaining variables, and the resulting variation function is summable as a function of these $n - 1$ variables. If f^i is also continuous as a function of x_i for almost all of the values of the remaining variables it is of type $\mathfrak{L}$. The set $\mathfrak{L}$ plays the part in more variables of the continuous functions of bounded variation in one variable. Finally, if continuity is replaced by absolute continuity in the definition of $\mathfrak{L}$, we obtain the Sobolev space W_1^1 which is the natural set to be taken as the analogue of the absolutely continuous functions.

If $f \in$ BVC, then clearly f is in W_1^1 if and only if the partial derivative of f are measures absolutely continuous with respect to Lebesgue n measure.

In order to treat the derivative measures for all $f \in$ BVC, and then the measures for those $f \in \mathfrak{L}$, it is convenient to obtain a lemma regarding sets of $n - 1$ dimensional integral geometric measure zero. Before doing this, we recall the meanings of the measures α_f and β_f associated with an $f \in$ BVC. The partial derivatives of f are a vector valued measure $(\mu_1, \ldots, \mu_n)$. With λ for Lebesgue n measure, the area measure $\alpha_f(E)$, of a Borel set E, is the total variation measure of the vector measure $(\lambda; \mu_1, \ldots, \mu_n)$. The coarea measure β_f is similarly associated with $(\mu_1, \ldots, \mu_n)$. The measure α_f may also be obtained as a lower semicontinuous

Received by the editors January 29, 1979 and, in revised form, April 18, 1979.
AMS (*MOS*) *subject classifications* (1970). Primary 26A24.
Key words and phrases. Hausdorff measure, integral geometric measure, rectifiable.
[1]This author's contribution supported in part by a grant from the NSF.
[2]This author's contribution made in part by a visit to the Department of Mathematics and of Statistics at Purdue University.

extension of an area functional to the set of summable functions from the set of piecewise linear functions, and then by extending this area from a function on rectangles to a measure on Borel sets. A general discussion of numerical valued measures of "area" type associated with vector valued measures may be found in [3] where further references are given.

LEMMA 1. *If $f \in BVC$ and S has $n-1$ dimensional integral geometric measure zero, then $\mu_i(S) = 0$, $i = 1, \dots, n$.*

PROOF. The projection of S has $n-1$ dimensional Lebesgue measure zero on almost all hyperplanes through the origin. There is an orthogonal coordinate system such that the projection of S on each coordinate hyperplane has $n-1$ dimensional Lebesgue measure zero. Let $\nu_1, \dots, \nu_n$ be the directional derivatives of f in these coordinate directions. By Krickeberg's lemma (see [3] for a proof), $\nu_i(S) = \int_{R^{n-1}} v_f^i(S, \bar{x})d\bar{x}$, where $\bar{x}$ varies over the space R^{n-1} determined by the $n-1$ coordinates left after the ith coordinate in this system is omitted, and $v_f^i(S, \bar{x})$ is the variation measure of f in the ith coordinate on the intersection of S with the line obtained by fixing the other coordinates at $\bar{x}$. By our choice of coordinates, $v_f^i(S, \bar{x}) = 0$ almost everywhere, so that $\nu_i(S) = 0$, $i = 1, \dots, n$. Moreover, $\lambda(S) = 0$, so our result follows from the inequalities

$$\lambda(S) + \nu_1(S) + \cdots + \nu_n(S) \geqslant \alpha_f(S) \geqslant \mu_i(S), \qquad i = 1, \dots, n.$$

As a corollary to this lemma we obtain

THEOREM 1. *If $f \in BVC$ and $H^{n-1}(S) = 0$, then $\mu_1(S) = \cdots = \mu_n(S) = 0$.*

PROOF. If S is of Hausdorff $n-1$ dimensional measure zero then it is of $n-1$ integral geometric measure zero.

Thus, if $f \in$ BVC then its partial derivative measures are absolutely continuous with respect to Hausdorff $n-1$ dimensional measure.

The case is more complicated for functions in $\mathfrak{L}$. We first give a fact which provides a setting for our main result. We restrict attention to the unit cube Q.

PROPOSITION 1. *For each $\alpha > n - 1 + p$, $0 < p < 1$, there is an $f \in \mathfrak{L}$ and $S \subset Q$ such that $H^\alpha(S) = 0$ and $\mu_1(S) > 0$.*

PROOF. Let $C \subset [0, 1]$ be a Cantor set of Hausdorff dimension less than p, and let f_1 be the Cantor function associated with C. Define f on Q by $f(x_1, \dots, x_n) = f_1(x_1)$. Let

$$S = C \times \underbrace{[0, 1] \times \cdots \times [0, 1]}_{n-1 \text{ copies}}.$$

It is a standard computation to show that $H^\alpha(S) = 0$ but that $\mu_1(S) = 1$.

Our main result, Theorem 2, makes use of the Federer structure theorem for sets of finite Hausdorff $n-1$ measure. It then uses the fact that a purely nonrectifiable set has integral geometric measure zero. A proof is given in [4, Theorems 6.9 and 6.11].

THEOREM 2. *If* $f \in \mathfrak{L}$ *and* S *is such that* $H^{n-1}(S) < \infty$ *then* $\mu_1(S) = \cdots = \mu_n(S) = 0$.

PROOF. By the Federer structure theorem, [**1**, p. 297], $S = Z \cup A$, where Z is purely nonrectifiable and A is countably rectifiable. By the above remark, Z has $n - 1$ dimensional integral geometric measure zero. By [**1**, p. 267], $A = N \cup (\bigcup B_k)$, where N has Hausdorff $n - 1$ dimensional measure zero and each B_k lies on an $n - 1$ dimensional manifold of class C^1. Each B_k is the union of countably many sets D_{km}, $(B_k = \bigcup D_{km})$, such that, for each D_{km}, there is an orthogonal coordinate system such that the set D_{km} lies on the graph of a real function on each coordinate hyperplane in this system. However, as is shown in [**2**], each $f \in \mathfrak{L}$ may be taken to be continuous on almost all lines in every direction (i.e., there is a function of this sort equal almost everywhere to f). If $\nu_1, \ldots, \nu_n$ are the measures which are the directional derivatives of f in these coordinate directions it follows from the continuity of f on almost all the lines and the fact that D_{km} has at most one point on each line in this direction, that $\nu_i(D_{km}) = 0$, $i = 1, \ldots, n$. Then $\alpha_f(D_{km}) = 0$. It follows that $\alpha_f(B_k) = 0$, $k = 1, 2, \ldots$, so that $\alpha_f(\bigcup B_k) = 0$. By Lemma 1, $\alpha_f(N) = \alpha_f(Z) = 0$. It follows that $\alpha_f(S) = 0$. Then $\mu_1(S) = \cdots = \mu_n(S) = 0$ and the theorem is proved.

REFERENCES

1. H. Federer, *Geometric measure theory*, Springer, New York, 1969.

2. C. Goffman, *A characterization of linearly continuous functions whose partial derivatives are measures*, Acta Math. **117** (1967), 165–190.

3. C. Goffman and J. B. Serrin, *Sublinear functions of measures and variational integrals*, Duke Math. J. **31** (1964), 159–178.

4. E. J. Mickie, *On a decomposition theorem of Federer*, Trans. Amer. Math. Soc. **92** (1959), 322–335.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 49707

DEPARTMENT OF MATHEMATICS, ACADEMIA SINICA, NANKANG, TAIPEI, TAIWAN