A THEOREM OF C. RYLL-NARDZEWSKI AND METRIZABLE L.C.A. GROUPS

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Abstract. \( \Gamma \) denotes a metrizable locally compact abelian group and \( \overline{\Gamma} \) its Bohr compactification. Let \( \gamma \in \Gamma \) be a cluster point of some subset \( E \) of \( \Gamma \) in the topology of \( \Gamma \). Then there are two disjoint subsets of \( E \) which also cluster at \( \gamma \) in the Bohr group topology. The proof is elementary and provides a new proof of the theorem of C. Ryll-Nardzewski on cluster points of \( \mathcal{I} \)-sets in \( \mathbb{R} \). Given the continuum hypothesis, either theorem characterizes metrizability in locally compact abelian groups. One of these characterizations is shown to be equivalent to the continuum hypothesis.

\( \Gamma \) denotes a locally compact abelian group whose dual is \( G \). \( \overline{\Gamma} \), the Bohr compactification of \( \Gamma \), is the l.c.a. group that is dual to \( G_d \), \( G \) with the discrete topology.

Theorem 1. Suppose that \( \Gamma \) is metrizable. Let \( E \) be a subset of \( \Gamma \) which clusters at \( \gamma \in \Gamma \) in the topology of \( \overline{\Gamma} \). Then there are two disjoint subsets of \( E \) which likewise cluster at \( \gamma \).

Proof. A basic neighborhood \( U = U(\gamma; g_1, \ldots, g_n; \varepsilon) \) of \( \gamma \) in \( \overline{\Gamma} \) is determined by a finite set of points \( g_1, \ldots, g_n \) from \( G \) and some \( \varepsilon > 0 \). Precisely, \( U \) consists of \( \lambda \) in \( \overline{\Gamma} \) such that \( |\lambda(g_i) - \gamma(g_i)| < \varepsilon \) for \( 1 \leq i \leq n \). Because \( \gamma \) is a cluster point of \( E \) in \( \overline{\Gamma} \), given any finite subset \( F \subseteq E \) there is some \( \lambda \in E \setminus F \) such that \( \lambda \in U \). Since both \( \lambda \) and \( \gamma \) are from \( \Gamma \) and thus continuous on \( G \) with respect to the original topology, there is some neighborhood \( V \) of \((g_1, \ldots, g_n)\) in \( G^n \) such that \( (h_1, \ldots, h_n) \in V \) implies \( \lambda \in U(\gamma; h_1, \ldots, h_n; 3\varepsilon) \). For any compact subset \( K \) of \( G \), \( K^n \) is covered by a finite number of such \( V \)'s. We may conclude that for \( K \) a compact subset of \( G \), for \( n > 0 \), for \( \varepsilon > 0 \), and \( F \) a finite subset of \( E \), there exist \( \lambda_1, \ldots, \lambda_m \) in \( E \setminus F \) such that \( (h_1, \ldots, h_n) \in K^n \) implies that for some \( 1 < j < m \), \( \lambda_j \in U(\gamma; h_1, \ldots, h_n; \varepsilon) \). Denote the set \( \{\lambda_1, \ldots, \lambda_m\} \) as \( D(K, n, e, F) \).

We now use the hypothesis that \( \Gamma \) is metrizable. This occurs exactly when \( G \) is \( \sigma \)-compact. Let \( G \) be \( \bigcup_{n=1}^{\infty} K_n \) with each \( K_n \) compact and \( K_n \subseteq K_{n+1} \). Define two sequences of finite subsets of \( E \) inductively. Set \( T_0 = S_0 = \emptyset \). Let \( S_{n+1} = D(K_{n+1}, n + 1, (n + 1)^{-1}, F_{n+1}) \) where \( F_{n+1} = \bigcup_{j < n} (T_j \cup S_j) \). Then \( T_{n+1} = D(K_{n+1}, n + 1, (n + 1)^{-1}, F_{n+1} \cup S_{n+1}) \). Clearly \( \bigcup_n T_n \) and \( \bigcup_n S_n \) are disjoint subsets of \( E \) which cluster at \( \gamma \) in \( \overline{\Gamma} \). \( \square \)
DEFINITION. A subset $E$ of $\Gamma$ is said to be an $I$-set if every bounded function on $E$ can be extended to $\overline{\Gamma}$ as a continuous function. Equivalently, $E \subset \Gamma$ is an $I$-set if every bounded function on $E$ is the restriction to $E$ of an almost periodic function on $\Gamma$ [1, p. 32].

The following result due to C. Ryll-Nardzewski [2] for $\Gamma = R$ is a corollary of Theorem 1.

COROLLARY. Let $\Gamma$ be a metrizable l.c.a. group. If $E \subset \Gamma$ is an $I$-set, then $E$ has no points of $\Gamma$ as cluster points in $\overline{\Gamma}$. Consequently the union of $E$ with any finite subset of $\Gamma$ is also an $I$-set.

The theorem and the corollary are both false when $G = T_d$, the circle with the discrete topology. In that case $\Gamma = \overline{\Gamma} = \mathbb{Z}$, the Bohr compactification of the integer group $\mathbb{Z}$. The set $E$ which consists of powers of 3 is an $I$-set, both as a subset of $\mathbb{Z}$ and of $\Gamma$. The set $E$ has uncountably many cluster points ($\Gamma = \overline{\Gamma}$). This example is characteristic of nonmetrizable groups if the continuum hypothesis is assumed; i.e., $2^{\aleph_0} = \aleph_1$. In what follows (CH) indicates use of the continuum hypothesis.

PROPOSITION (CH). Let $\Gamma$ be a nondegenerate compact abelian group, and let $I$ be an uncountable set. Then there is an embedding of the Stone-Cech compactification $\beta(\mathbb{N})$ in the group $\Gamma'$, where $\mathbb{N}$ is the set of natural numbers.

PROOF. Let 2 be the discrete two-point space. Since $\mathbb{N}$ is discrete, the evaluation map $\text{ev}: \mathbb{N} \to 2^\mathbb{N}$ extends to an embedding of $\beta(\mathbb{N})$ into $2^\mathbb{N}$. Since $I$ is uncountable, there is an injection of $2^\mathbb{N}$ into $I$, and this then allows an embedding of $2^\mathbb{N}$ into $\Gamma'$, since $\Gamma$ is nondegenerate and compact. The composition of the embedding of $\beta(\mathbb{N})$ into $2^\mathbb{N}$ with this embedding into $\Gamma'$ then yields the desired embedding. □

LEMMA 1. Let $\Gamma$ and $\Gamma'$ be locally compact abelian groups, and let $f: \Gamma \to \Gamma'$ be a continuous surmorphism of $\Gamma$ onto $\Gamma'$. Then any $I$-set in $\Gamma'$ lifts to an $I$-set in $\Gamma$.

PROOF. Let $E$ be an $I$-set in $\Gamma'$. For each point of $E$, choose a point in $\Gamma$ which maps onto it under $f$, and call the resulting set $F$. Then $f|F: F \to E$ is a bijection. Since $f$ is a continuous surmorphism, $f$ extends to a continuous surmorphism $\tilde{f}: \overline{\Gamma} \to \overline{\Gamma'}$.

Now, let $g: F \to C$ be any bounded function. Then $g \cdot f^{-1}$ is a bounded function on the $I$-set $E$, and so this extends to a continuous function $g_1: \overline{\Gamma'} \to C$. The composition $g_1 \cdot \tilde{f}: \overline{\Gamma} \to C$ then provides the desired extension of $g$ to $\overline{\Gamma}$. Thus $F$ is an $I$-set in $\Gamma$. □

Below $I^H$ and $H^I$ denote direct sums and direct products of the group $H$, respectively.

LEMMA 2. Let $G$ be an uncountable abelian group. Then $G$ has a subgroup $G'$ which is isomorphic to $I^H$, where $H$ is a nondegenerate group, and $I$ is an uncountable set (of the same cardinality as $G$).
Proof. Let $D$ be the divisible hull of $G$, i.e., $D$ is a minimal divisible extension of $G$. Then $D$ can be represented as $(\mathbb{Q}^\omega \oplus \bigoplus_{p \in P} \mathbb{Z}(p^\infty))$ where $\mathbb{Q}$ is the group of rationals, $r_0$ is the torsion-free rank of $G$, and $r_p$ is the $p$-rank of $G$ for each prime $p$ in $P$, the set of primes [3, Appendix A, pp. 444–446]. Now, $D$ is uncountable since $G$ is (in fact of equal cardinality), and so either $r_0$ is uncountable, or $r_p$ is uncountable for some prime $p$. In any event, we conclude that $G$ has an uncountable set of independent elements all having the same order. The group $G'$ that this set generates is then isomorphic to $I\cdot H$, where $H$ is the cyclic group of the order in question, and $I$ is the set of independent elements of the order.

Theorem 2 below is a converse of Theorem 1 under the continuum hypothesis. Theorems 1 and 2 together characterize metrizable locally compact abelian groups under the continuum hypothesis assumption.

Theorem 2 (CH). Let $\Gamma$ be a nonmetrizable l.c.a. group. Then there is an infinite $I$-set $E \subset \Gamma$ whose closure in $\Gamma$ is a subset of $\Gamma$. Consequently, Theorem 1 and its corollary are false for $\Gamma$.

Proof. Suppose that $\Gamma$ is a nonmetrizable locally compact abelian group. Then, according to the Principle Structure Theorem for locally compact abelian groups [1, p. 40], $\Gamma$ has an open subgroup which is a direct product of a vector group $V$ and a compact subgroup $\Gamma'$ of $\Gamma$. Since $V$ is metrizable and $\Gamma$ is not, and since $V \times \Gamma'$ is open, it follows that $\Gamma'$ is also not metrizable. Now, Tietze’s Extension Theorem implies that any $I$-set in $\Gamma'$ is also an $I$-set in $\Gamma$. Moreover, since $\Gamma'$ is compact and the map from $\Gamma$ to $\bar{\Gamma}$ is continuous, the image of $\Gamma'$ in $\bar{\Gamma}$ is closed, and so any $I$-set in $\Gamma'$ has its closure a subset of $\Gamma'$, and hence of the image of $\Gamma$ in $\bar{\Gamma}$. This shows that it suffices to produce the desired $I$-set in $\Gamma'$, or said another way, it suffices to consider the case when $\Gamma$ is compact.

Given that $\Gamma$ is a compact nonmetrizable abelian group, its character group $G$ is then an uncountable abelian group. Thus, Lemma 2 implies that $G$ has a subgroup $G' \simeq I\cdot H$, where $H$ is a nondegenerate group, and $I$ is uncountable. Then, the character group $\Gamma'$ of $G'$ is a quotient of $\Gamma$, and is isomorphic to $\Delta'$, where $\Delta$ is the character group of $H$. Lemma 1 implies that it suffices to produce the desired $I$-set in $\Gamma'$. Moreover, the proposition implies that $\beta(N)$ is embedded in $\Gamma'$, and this clearly makes the corresponding copy of $N$ an $I$-set in $\Gamma'$, and we are done.

The existence of an infinite $I$-set in an arbitrary nonmetrizable l.c.a. group is actually equivalent to the continuum hypothesis. To see this, let $G$ be a discrete group whose cardinality is the first uncountable cardinal and let $E$ be an infinite $I$-set in $G^\wedge = \Gamma$. For each partition of the set $E$ into two disjoint subsets, choose a finitely-supported discrete measure with rational coefficients whose Fourier-Stieltjes transform is approximately 1 on one subset of the partition of $E$ and approximately 0 on the other subset of $E$. The cardinality of the set of such discrete measures is the same as that of $G$ but the cardinality of possible partitions of $E$ is at least $2^{2^{2^\omega}}$. Thus, $2^{2^{2^\omega}} \leq \aleph_1$, i.e. the continuum hypothesis holds.

It is also false (without CH) that the theorem of C. Ryll-Nardzewski is characteristic of metrizable l.c.a. groups. This follows from the preceding observation.
Otherwise we could have a compact, nonmetrizable group in which every $I$-set were finite and hence the Ryll-Nardzewski theorem would hold for that nonmetrizable group.

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References