$l^\infty/c_0$ HAS NO EQUIVALENT STRICTLY CONVEX NORM

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Abstract. It is shown that the quotient space $l^\infty/c_0$ does not admit an equivalent strictly convex norm.

Introduction. We say that a normed space $X$, $||\ ||$ is strictly convex provided $||x + y|| < 2$ whenever $||x|| = ||y|| = 1$ and $x \neq y$. This means also that any member $x^*$ in the dual $X^*$ of $X$ ($x^* \neq 0$) attains its norm in at most one point of the unit ball of $X$. A space is called strictly convexifiable if there exists an equivalent strictly convex norm on the space.

It is known that $l^\infty = l^\infty(N)$ is strictly convexifiable (cf. [1], [2]). However the purpose of this note is to prove that the quotient $l^\infty/c_0$ fails this property, a problem raised independently by J. J. Schäffer and J. Diestel [3].

We will denote infinite subsets of the integers $N$ by letters $L$, $M$, $N$, $P$. If $L$ is an infinite subset of $N$, let $P_N(L)$ be the set of all infinite subsets of $L$. Our proof is based on the following elementary result.

Lemma. Suppose $x^* \in (l^\infty)^*$, $L \in P_\infty(N)$ and $\epsilon > 0$. Then there is some $M \in P_N(L)$ such that $|x^*(x)| < \epsilon$ whenever $x \in l^\infty$, $||x|| = 1$ and $x_n = 0$ if $n \notin M$.

Proof. We may, of course, assume $||x^*|| = 1$. Take an integer $d > \epsilon^{-1}$ and disjoint members $M_i (1 < i < d)$ of $P_N(L)$.

If each $M_i$ fails the property, then we find elements $x^{(i)} (1 < i < d)$ in $l^\infty$, so that:

1. $||x^{(i)}|| = 1$,
2. $x^{(i)}_n = 0$ if $n \notin M_i$, and
3. $x^*(x^{(i)}) > \epsilon$.

Consider now the vector $x = x^{(1)} + x^{(2)} + \cdots + x^{(d)}$. Obviously $||x|| = 1$ and $x^*(x) > d$. This is the required contradiction.

We are now ready to prove the following theorem.

Theorem. Let $||\ ||$ be an equivalent norm on $l^\infty/c_0$. Then $||\ ||$ is not strictly convex.

Proof. Let $Y = l^\infty/c_0$, $||\ ||$ and $\pi: l^\infty \to Y$ the quotient map be given. Let $(\epsilon_i)$ be a sequence of positive numbers converging to $0$. We make the following construction: Take $F_1 = \{x \in l^\infty; ||x|| < 1\}$ and $s_1 = \sup\{||\pi(x)||; x \in F_1\}$. Let $x^{(1)} \in F_1$ be such that $||\pi(x^{(1)})|| > s_1 - \epsilon_1$ and

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consider \( y_1^* \in Y^* \), \( \| y_1^* \| = 1 \) with \( y_1^* \pi(x^{(1)}) > s_1 - \epsilon_1 \).

Since \( \pi^*(y_1^*) \in (l^\infty)^* \), application of the lemma provides \( L_1 \in P_\infty(N) \) such that \( |y_1^* \pi(x)| < \epsilon_1 \) if \( x \in l^\infty \), \( x < 1 \) and \( x_n = 0 \) for \( x \not\in L_1 \). Take \( F_2 = \{ x \in F_1; x_n = x_n^{(1)} \text{ for } n \not\in L_1 \} \) and \( s_2 = \sup\{ \|\pi(x)\|; x \in F_2 \} \). Let \( x^{(2)} \in F_2 \) be such that \( \|\pi(x^{(2)})\| > s_2 - \epsilon_2 \) and consider \( y_2^* \in Y^* \), \( \| y_2^* \| = 1 \) with \( y_2^* \pi(x^{(2)}) > s_2 - \epsilon_2 \). Again by the lemma, we get \( L_2 \in P_\infty(L_1) \) such that \( |y_2^* \pi(x)| < \epsilon_2 \) for \( x \in l^\infty \), \( x < 1 \) and \( x_n = 0 \) if \( n \not\in L_2 \).

In general \( F_{i+1} = \{ x \in F_i; x_n = x_n^{(i)} \text{ for } n \not\in L_i \} \) and \( s_{i+1} = \sup\{ \|\pi(x)\|; x \in F_{i+1} \} \). Let \( x^{(i+1)} \in F_{i+1} \) satisfy \( \|\pi(x^{(i+1)})\| > s_{i+1} - \epsilon_{i+1} \) and take \( y_{i+1}^* \in Y^* \), \( \| y_{i+1}^* \| = 1 \) with \( y_{i+1}^* \pi(x^{(i+1)}) > s_{i+1} - \epsilon_{i+1} \). By the lemma, there is some \( L_{i+1} \in P_\infty(L_i) \) so that \( |y_{i+1}^* \pi(x)| < \epsilon_{i+1} \) if \( x \in l^\infty \), \( x < 1 \) and \( x_n = 0 \) for \( n \not\in L_{i+1} \).

Since for \( x \in F_{i+1} \) we have \( \|x - x^{(i)}\| < 2 \) and \( x_n - x_n^{(i)} = 0 \) for \( n \not\in L_i \), it follows that \( |y_i^* \pi(x - x^{(i)})| < 2\epsilon_i \) and thus \( y_i^* \pi(x) > y_i^* \pi(x^{(i)}) - 2\epsilon_i > s_i - 3\epsilon_i \).

Clearly \( (F_i) \) is a decreasing sequence of nonvoid \( o(l^\infty, l^1) \) compact subsets of \( l^\infty \). Also \( (s_i) \) decreases and we let \( s = \lim_{i \to \infty} s_i \). Take some element \( x^{(\infty)} \) in \( \bigcap F_i \) and let \( y^* \) be a \( \omega^* \)-cluster point of \( (y_i^*) \). We consider the subset \( S = \bigcap \pi(F_i) \) of \( Y \). For a fixed \( y \in S \), we find \( \|y\| \leq s \).

Since \( y \in \pi(F_{i+1}) \), it follows that \( y_i^*(y) > s_i - 3\epsilon_i \geq s - 3\epsilon_i \) for all \( i \). Consequently \( y^*(y) > s \) and thus \( y^*(y) = s = \|y\| \). We show that \( S \) contains more than one point. In particular, this will imply that \( s > 0 \) and \( \|y^*\| = 1 \). So the proof will be complete. Because \( (L_i) \) is decreasing in \( P_\infty(N) \), there is some \( L \in P_\infty(N) \) with \( L \setminus L_i \) finite for all \( i \). Assume \( x \in l^\infty \), \( \|x\| = 1 \) and \( x_n = x_n^{(\infty)} \) for \( n \not\in L \). It is possible to take \( \pi(x) \neq \pi(x^{(\infty)}) \) since \( L \) is infinite. We claim that \( \pi(x) \in S \). To see this, fix some \( i \) and remark that there is \( x' \in l^\infty \), \( \|x'\| = 1 \), \( x'_n = x_n \) for \( n \not\in L \setminus L_i \) and \( x'_n = x_n^{(\infty)} \) for \( n \not\in L_i \) (\( x' \) depends of course on \( i \)). Thus \( x'_n = x_n^{(\infty)} = x_n^{(i)} \) if \( n \not\in L_i \), for all \( j = 1, \ldots, i - 1 \). Now \( x' \in F_i \) and proceeding by induction we see that \( x' \in F_j \) (\( 1 < j < i \)). Because \( L \setminus L_i \) is finite, \( \pi(x) = \pi(x') \in \pi(F_i) \). Consequently \( \pi(x) \in \bigcap \pi(F_i) \), which is what must be obtained.

**References**

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