$l^\infty/c_0$ HAS NO EQUIVALENT STRICTLY CONVEX NORM

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**Abstract.** It is shown that the quotient space $l^\infty/c_0$ does not admit an equivalent strictly convex norm.

**Introduction.** We say that a normed space $X$, $\| \|$ is strictly convex provided $\|x + y\| < 2$ whenever $\|x\| = \|y\| = 1$ and $x \neq y$. This means also that any member $x^*$ in the dual $X^*$ of $X$ ($x^* \neq 0$) attains its norm in at most one point of the unit ball of $X$. A space is called strictly convexifiable if there exists an equivalent strictly convex norm on the space.

It is known that $l^\infty = l^\infty(N)$ is strictly convexifiable (cf. [1], [2]). However the purpose of this note is to prove that the quotient $l^\infty/c_0$ fails this property, a problem raised independently by J. J. Schäffer and J. Diestel [3].

We will denote infinite subsets of the integers $\mathbb{N}$ by letters $L, M, N, P$. If $L$ is an infinite subset of $\mathbb{N}$, let $P_\infty(L)$ be the set of all infinite subsets of $L$. Our proof is based on the following elementary result.

**Lemma.** Suppose $x^* \in (l^\infty)^*$, $L \in P_\infty(\mathbb{N})$ and $\epsilon > 0$. Then there is some $M \in P_\infty(L)$ such that $|x^*(x)| < \epsilon$ whenever $x \in l^\infty$, $\|x\| = 1$ and $x_n = 0$ if $n \notin M$.

**Proof.** We may, of course, assume $\|x^*\| = 1$. Take an integer $d > \epsilon^{-1}$ and disjoint members $M_i (1 < i < d)$ of $P_\infty(L)$.

If each $M_i$ fails the property, then we find elements $x(i)$ ($1 < i < d$) in $l^\infty$, so that:

1. $\|x(i)\| = 1$,
2. $x_m(i) = 0$ if $n \notin M_i$, and
3. $x^*(x(i)) > \epsilon$.

Consider now the vector $x = x(1) + x(2) + \cdots + x(d)$. Obviously $\|x\| = 1$ and $x^*(x) > d$. This is the required contradiction.

We are now ready to prove the following theorem.

**Theorem.** Let $\| \|$ be an equivalent norm on $l^\infty/c_0$. Then $\| \|$ is not strictly convex.

**Proof.** Let $Y = l^\infty/c_0$, $\| \|$ and $\pi: l^\infty \to Y$ the quotient map be given. Let $(\epsilon_i)$ be a sequence of positive numbers converging to 0. We make the following construction: Take $F_i = \{x \in l^\infty; \|x\| < 1\}$ and $s_i = \sup\{\|\pi(x)\|; x \in F_i\}$. Let $x(i) \in F_i$ be such that $\|\pi(x(i))\| > s_i - \epsilon_i$ and

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consider $y_1^* \in Y^*$, $\|y_1^*\| = 1$ with $y_1^* \pi(x^{(1)}) > s_1 - \epsilon_1$.

Since $\pi^*(y_1^*) \in (l^{\infty})^*$, application of the lemma provides $L_1 \in P_{\infty}(N)$ such that $|y_1^* \pi(x)| < \epsilon_1$ if $x \in l^{\infty}$, $x < 1$ and $x_n = 0$ for $x \not\in L_1$. Take $F_2 = \{x \in F_1; x_n = x_n^{(1)}$ for $n \not\in L_1\}$ and $s_2 = \sup\{\|\pi(x)\|; x \in F_2\}$. Let $x^{(2)} \in F_2$ be such that $\|\pi(x^{(2)})\| > s_2 - \epsilon_2$ and consider $y_2^* \in Y^*$, $\|y_2^*\| = 1$ with $y_2^* \pi(x^{(2)}) > s_2 - \epsilon_2$. Again by the lemma, we get $L_2 \in P_{\infty}(L_1)$ such that $|y_2^* \pi(x)| < \epsilon_2$ for $x \in l^{\infty}$, $x < 1$ and $x_n = 0$ if $n \not\in L_2$.

In general $F_{i+1} = \{x \in F_i; x_n = x_n^{(i)}$ for $n \not\in L_i\}$ and $s_{i+1} = \sup\{\|\pi(x)\|; x \in F_{i+1}\}$. Let $x^{(i+1)} \in F_{i+1}$ satisfy $\|\pi(x^{(i+1)})\| > s_{i+1} - \epsilon_{i+1}$ and take $y_{i+1}^* \in Y^*$, $\|y_{i+1}^*\| = 1$ with $y_{i+1}^* \pi(x^{(i+1)}) > s_{i+1} - \epsilon_{i+1}$. By the lemma, there is some $L_{i+1} \in P_{\infty}(L_i)$ so that $|y_{i+1}^* \pi(x)| < \epsilon_{i+1}$ if $x \in l^{\infty}$, $x < 1$ and $x_n = 0$ for $n \not\in L_{i+1}$.

Since for $x \in F_{i+1}$ we have $\|x - x^{(i)}\| < 2$ and $x_n - x_n^{(0)} = 0$ for $n \not\in L_i$, it follows that $|y_i^* \pi(x - x^{(i)})| < 2 \epsilon_i$ and thus $y_i^* \pi(x) > y_i^* \pi(x^{(i)}) - 2 \epsilon_i > s_i - 3 \epsilon_i$.

Clearly $(F_i)$ is a decreasing sequence of nonvoid $\sigma(l^{\infty}, l^1)$ compact subsets of $l^{\infty}$. Also $(s_i)$ decreases and we let $s = \lim_{i \to \infty} s_i$. Take some element $x^{\infty}$ in $\cap F_i$ and let $y^*$ be a $\omega^*$-cluster point of $(y_i^*)$. We consider the subset $S = \cap \pi(F_i)$ of $Y$. For a fixed $y \in S$, we find $\|y\| < s$.

Since $y \in \pi(F_{i+1})$, it follows that $y_i^*(y) > s_i - 3 \epsilon_i > s - 3 \epsilon_i$ for all $i$. Consequently $y^*(y) > s$ and thus $y^*(y) = s = \|y\|$. We show that $S$ contains more than one point. In particular, this will imply that $s > 0$ and $\|y^*\| = 1$. So the proof will be complete. Because $(L_i)$ is decreasing in $P_{\infty}(N)$, there is some $L \in P_{\infty}(N)$ with $L \setminus L_i$ finite for all $i$. Assume $x \in l^{\infty}$, $\|x\| = 1$ and $x_n = x_n^{\infty}$ for $n \not\in L$. It is possible to take $\pi(x) \neq \pi(x^{\infty})$ since $L$ is infinite. We claim that $\pi(x) \in S$. To see this, fix some $i$ and remark that there is $x' \in l^{\infty}$, $\|x'\| = 1$, $x_n' = x_n$ for $n \not\in L \setminus L_i$, and $x_n' = x_n^{\infty}$ for $n \not\in L_i$ ($x'$ depends of course on $i$). Thus $x_i' = x_n^{\infty} = x_n^{(i)}$ if $n \not\in L_j$, for all $j = 1, \ldots, i - 1$. Now $x' \in F_i$ and proceeding by induction we see that $x' \in F_j (1 < j < i)$. Because $L \setminus L_i$ is finite, $\pi(x) = \pi(x') \in \pi(F_i)$. Consequently $\pi(x) \in \cap \pi(F_i)$, which is what must be obtained.

References


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