ON $L^1$ ISOMORPHISMS

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Abstract. Let $(X_1, \Sigma_1, \mu_1)$ and $(X_2, \Sigma_2, \mu_2)$ be two $\sigma$-finite measure spaces. We show that any isomorphism $T$ of the Banach space $L^1(X_1, \Sigma_1, \mu_1)$ onto the Banach space $L^1(X_2, \Sigma_2, \mu_2)$ which satisfies $\|T\| \cdot \|T^{-1}\| < 2$ induces a transformation of the underlying measure spaces.

In [1] and [2] it has been shown by D. Amir and M. Cambern that if $Y_1$ and $Y_2$ are compact Hausdorff spaces, and if there exists an isomorphism $T$ of $C(Y_1)$ onto $C(Y_2)$ with $\|T\| \cdot \|T^{-1}\| < 2$, then $Y_1$ and $Y_2$ are homeomorphic. In this note, we use this theorem to prove an analogous result for $L^1$ spaces. (Concerning the terminology "regular set isomorphism" as it is used in this paper, the reader is referred to [6].)

Theorem. Let $(X_1, \Sigma_1, \mu_1)$ and $(X_2, \Sigma_2, \mu_2)$ be $\sigma$-finite measure spaces. If there exists an isomorphism $T$ of $L^1(X_1, \Sigma_1, \mu_1)$ onto $L^1(X_2, \Sigma_2, \mu_2)$ satisfying $\|T\| \cdot \|T^{-1}\| < 2$, then there exists a regular set isomorphism $\Phi$ of $(X_1, \Sigma_1, \mu_1)$ onto $(X_2, \Sigma_2, \mu_2)$.

Proof. Since the measure spaces are $\sigma$-finite, the dual space of $L^1(X_i, \Sigma_i, \mu_i)$ is $L^\infty(X_i, \Sigma_i, \mu_i)$, $i = 1, 2$ [4, p. 289]. Hence the adjoint transformation $T^*$ is an isomorphism of $L^\infty(X_2, \Sigma_2, \mu_2)$ onto $L^\infty(X_1, \Sigma_1, \mu_1)$ satisfying $\|T^*\| \cdot \|T^{-1}\| < 2$. Now $L^\infty(X_i, \Sigma_i, \mu_i)$ is isometrically isomorphic to $C(Y_i)$, $i = 1, 2$, under the map $\rho_i(f) = \hat{f}$, where $Y_i$ is the maximal ideal space of $L^\infty(X_i, \Sigma_i, \mu_i)$ and $\rho_i$ is the Gelfand representation of $L^\infty(X_i, \Sigma_i, \mu_i)$, ([4, p. 445] or [5, p. 17]). Define a map $R$ of $C(Y_2)$ to $C(Y_1)$ by $R(\hat{f}) = \rho_1 \circ T^* \circ \rho_2^{-1}(\hat{f})$, for $\hat{f} \in C(Y_2)$. Then clearly $R$ is an isomorphism of $C(Y_2)$ onto $C(Y_1)$ with $\|R\| \cdot \|R^{-1}\| < 2$.

It thus follows that there exists a homeomorphism $\tau$ mapping $Y_1$ onto $Y_2$. And, being a homeomorphism, $\tau$ carries the clopen sets of $Y_1$ onto the clopen sets of $Y_2$. Now if $A_i \in \Sigma_i$, then $\chi_{A_i}$ is the characteristic function of a clopen subset $U_{A_i}$ of $Y_i$, and every clopen subset $U$ of $Y_i$ is of the form $U_{A_i}$, for some $A_i \in \Sigma_i$, $i = 1, 2$ [5, p. 17]. Let $\Phi$ be the map from $\Sigma_1$ to $\Sigma_2$, defined modulo null sets by $\Phi(A_i) = A_2$ if $\tau(U_{A_i}) = U_{A_2}$, where $U_{A_i} \subseteq Y_i$ and is related to $A_i \in \Sigma_i$ as in the previous sentence.

If $\mathcal{N}_i$ denotes the family of null sets in $\Sigma_i$, then for $i = 1, 2$, $\Sigma_i/\mathcal{N}_i$ is isomorphic as a Boolean algebra with the clopen subsets of $Y_i$, under the correspondence $A_i \leftrightarrow U_{A_i}$. Moreover, the Boolean supremum of a sequence $U_{A_i}$ of clopen sets is the topological closure of the point set union of the $U_{A_i}$. Then, since the homeomorphism $\tau$ of $Y_1$ onto $Y_2$ preserves both point set unions and topological closures, it clearly effects an order isomorphism between the Boolean algebras of clopen sets.

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of the $Y$, from which it readily follows that the map $\Phi$ of the previous paragraph is a regular set isomorphism.

**Remarks and Problems.**

(a) The condition $\|T\| \|T^{-1}\| < 2$ in our theorem cannot be removed to allow for arbitrary isomorphisms of $L^1(X_1, \Sigma_1, \mu_1)$ onto $L^1(X_2, \Sigma_2, \mu_2)$ as the following example shows. Let $(X_1, \Sigma_1, \mu_1)$ be the measure space where $X_1 = [0, 1]$, $\Sigma_1$ is the \(\sigma\)-field of Lebesgue subsets of $[0, 1]$ and $\mu_1$ is Lebesgue measure. Let $(X_2, \Sigma_2, \mu_2)$ be defined as follows: $X_2 = [0, 1] \cup \{2\}$, $\Sigma_2$ consists of the Lebesgue measurable subsets of $X_2$, and $\mu_2$ is the sum of Lebesgue measure on $\Sigma_2$ and of the unit point mass concentrated at 2. For each $k = 0, 1, 2, \ldots$, let $I_k$ be the subset of $[0, 1]$ defined by $I_k = [(2^k - 1)/2^k, (2^k + 1 - 1)/2^k+1)$. We define a map $T$ from $L^1(X_1, \Sigma_1, \mu_1)$ to $L^1(X_2, \Sigma_2, \mu_2)$ by

\[
(T(f))(2) = \int_0^1 f(t) \, dt
\]

and

\[
(T(f))(t) = f(t) - 2^{k+1} \int_{I_k} f(t) \, dt + 2^{k+1} \int_{I_{k+1}} f(t) \, dt
\]

for $f \in L^1(X_1, \Sigma_1, \mu_1)$ and $t \in I_k$. $(T(f))$ has not been defined at 1, but since we are actually defining a map of equivalence classes rather than functions, the value of $(T(f))(1)$ is of no concern.) It is clear that $T$ is linear. It is moreover one-one and surjective since given $g \in L^1(X_2, \Sigma_2, \mu_2)$, the element $f \in L^1(X_1, \Sigma_1, \mu_1)$ defined by

\[
f(t) = g(t) - 2 \int_0^t g(t) \, dt + 2 \cdot g(2)
\]

for $t \in I_0$, and

\[
f(t) = g(t) - 2^{k+1} \int_{I_k} g(t) \, dt + 2^{k+1} \int_{I_{k+1}} g(t) \, dt
\]

for $t \in I_k$, $k > 0$, is such that $T(f) = g$. Thus $T$ is a continuous isomorphism of $L^1(X_1, \Sigma_1, \mu_1)$ onto $L^1(X_2, \Sigma_2, \mu_2)$. However, since $(X_2, \Sigma_2, \mu_2)$ contains an atom while $(X_1, \Sigma_1, \mu_1)$ does not, there can exist no regular set isomorphism of $(X_1, \Sigma_1, \mu_1)$ onto $(X_2, \Sigma_2, \mu_2)$.

(b) It is known (see [3]) that for the theorem mentioned in the first paragraph of this article, 2 can be replaced by no larger number in the condition $\|T\| \|T^{-1}\| < 2$. Is 2 also the "best" number for a theorem of the type obtained in this paper?

(c) Can a theorem analogous to the one of this article be established for $L^p$, $1 < p < \infty$, $p \neq 2$?

**References**


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