SYSTEMS OF PARTIAL DIFFERENTIAL OPERATORS
WITH FUNDAMENTAL SOLUTIONS
SUPPORTED BY A CONE

KIRK E. LANCASTER AND BENT E. PETERSEN

Abstract. Necessary and sufficient conditions are given for a system of partial
differential operators to have a fundamental solution supported by a convex salient
cone. As a simple application an overdetermined Cauchy problem is solved.

If A is a subset of \( \mathbb{R}^n \) and \( \mathcal{F} \) is a space of distributions on \( \mathbb{R}^n \) we denote by \( \mathcal{F}_A \)
the space of distributions in \( \mathcal{F} \) which have supports contained in \( A \). We denote by
\( \mathcal{D}' \) the space of all distributions on \( \mathbb{R}^n \), by \( \mathcal{S}' \) the space of temperate distributions,
and by \( \mathcal{S} \) the space of infinitely differentiable functions on \( \mathbb{R}^n \). If \( \Gamma \) is a closed
convex cone in \( \mathbb{R}^n \) with vertex at the origin, we denote by \( \Gamma^+ \) the dual cone defined
by \( \Gamma^+ = \{ \xi \in \mathbb{R}^n \mid \langle \xi, x \rangle > 0, x \in \Gamma \} \). Then \( \Gamma^{++} = \Gamma \). The interior \( \Gamma_0^+ \) of \( \Gamma^+ \) is
nonempty if and only if \( \Gamma \) is salient, i.e. contains no subspace other than \( \{0\} \). If \( \Gamma \) is
salient then \( \mathcal{D}'_{\Gamma} \) is a commutative ring relative to convolution. If \( H \) is a closed
half-space with interior normal \( \eta \in \Gamma_0^+ \) then \( \mathcal{D}'_H \) is a \( \mathcal{D}'_{\Gamma} \)-module, and differentiation
commutes with convolution in the usual fashion. Finally we note \( \mathcal{S}'_{\Gamma} \) is a
subring of \( \mathcal{D}'_{\Gamma} \). This fact is proved in the appendix below.

Let \( P(z) \) be a \( p \times q \) matrix over \( C[z_1, \ldots, z_n] \) and denote by \( P(D) \) the system of
partial differential operators obtained by replacing \( z_j \) in \( P(z) \) by \( \partial/\partial x_j \). If \( p < q \)
then a fundamental solution for \( P(D) \) is a \( q \times p \) matrix \( K \) over \( \mathcal{D}' \) such that

\[
P(D)K = \delta I
\]

where \( I \) is the \( p \times p \) identity matrix and \( \delta \) is the Dirac measure at 0. In case \( p = q \)
then \( P(D) \) has a fundamental solution with support in the closed convex salient
cone \( \Gamma \) if and only if \( P(D) \) is hyperbolic with respect to each direction in \( \Gamma_0^+ \), [1]. In
case \( p = q = 1 \) then \( P(D) \) has a temperate fundamental solution with support in
the closed convex salient cone \( \Gamma \) if and only if \( P(z) \neq 0 \) for each \( z \) in \( \Gamma_0^+ + i\mathbb{R}^n \).
This fact may be proved by means of an elementary inequality for polynomials, as
is done in the introduction to [9]. The temperate case with \( p = 1, q > 1 \) is also
considered in ([8], [9]) and may easily be generalized as is done below. In this note
we will give a sufficient, and in case \( \Gamma \) is semialgebraic, necessary condition for
\( P(D) \) in the case \( p < q \) to have a fundamental solution \( K \) with support in the
closed convex salient cone \( \Gamma \). Our methods do not apply in the nonsalient case. The
scalar case \( p = q = 1 \) with \( \Gamma \) nonsalient has been considered by A. Enqvist in [3]
and in the temperate case in [4]. We will prove the following two theorems.


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Theorem 1. Let $\Gamma$ be a closed convex salient cone and $p < q$. Then $P(D)$ admits a temperate fundamental solution $K$ with support in $\Gamma$ if and only if for each $z$ in the tube $\Gamma_0^+ + i\mathbb{R}^n$ the matrix $P(z)$ has rank $p$.

Theorem 2. Let $\Gamma$ be a closed convex salient cone and $p < q$. Then $P(D)$ admits a fundamental solution $K$ with support in $\Gamma$ if, and in the case $\Gamma$ is semialgebraic, only if there exists a convex open set $U$ in $\Gamma_0^+$ such that $tU \subseteq U$ for $t > 1$, $\Gamma_0^+ = \bigcup tU$ ($t > 0$) and for each $z$ in the tube $U + i\mathbb{R}^n$ the matrix $P(z)$ has rank $p$.

In the case $p = q$ we may dispense with the hypothesis that $\Gamma$ is semialgebraic. There are at least two ways to do this. If $P(D)$ has a fundamental solution with support in $\Gamma$ then the determinant $\det P(D)$ is hyperbolic with respect to each direction in $\Gamma^+$. From the theory of scalar hyperbolic operators [5] it follows that $\det P(D)$ is hyperbolic with respect to each direction in an open convex semialgebraic cone which contains $\Gamma_0^+$. By the lemma below we then obtain a closed convex semialgebraic cone $\Gamma' \subseteq \Gamma$ such that $P(D)$ has a fundamental solution with support in $\Gamma'$. Alternately, if $\Gamma$ is not assumed semialgebraic a modification of the proof of necessity produces an open set $U$ with the required properties other than convexity. In the case $p = q$, S. Bochner's theorem on tubes [6, Theorem 2.5.10] then shows we may replace $U$ by its convex hull.

We first reduce the $p \times q$ system to a $1 \times N$ system, $N = \binom{q}{p}$. The notation $|J| = p$ will mean that $J = (j_1, \ldots, j_p)$ where the $j_k$ are integers and $1 < j_k < q$ for each $k$. For each such $J$ let $P^J(z)$ be the $p \times p$ matrix whose $k$th column is the $j_k$th column of $P(z)$ and let $Q^J(z)$ be the determinant of $P^J(z)$.

Lemma. Let $\Gamma$ be a closed convex salient cone and $p < q$. Then $P(D)$ admits a fundamental solution (respectively, a temperate fundamental solution) with support in $\Gamma$ if and only if there exist distributions (respectively, temperate distributions) $\mathcal{L}_J$, $|J| = p$, with supports in $\Gamma$ such that

$$\sum_{|J| = p} Q^J(D) \mathcal{L}_J = \delta. \tag{2}$$

Here the prime over the summation symbol indicates that we sum only over $p$-indices $J = (j_1, \ldots, j_p)$ with $1 < j_1 < \cdots < j_p < q$. For the proof, suppose first that (2) holds with supp $L_j \subseteq \Gamma$. Let $Q^i_k(z)$ be the $(i, k)$-cofactor of $P^J(z)$, that is $(-1)^{i+k}$ times the determinant of the matrix obtained from $P^J(z)$ by removing the $i$th row and the $k$th column. Then

$$\sum_{h=1}^p P_{ih}(z) Q^i_h(z) = \begin{cases} Q^i_j(z) & \text{if } l = i, \\ 0 & \text{if } l \neq i, \end{cases}$$

where $J = (j_1, \ldots, j_p)$. If we set

$$K_{jl} = \sum_{h=1}^p \sum_{j' = 1}^q Q^{j'}_h(D) L_{j'}, \quad 1 < j < q, 1 < l < p,$$

where the inner sum is over $|J| = p$ such that $j_h = j$, then supp $K_{jl} \subseteq \Gamma$ and

$$\sum_{j=1}^q P_j(D) K_{jl} = \sum_{|J| = p} \sum_{h=1}^p P_{ih}(D) Q^i_h(D) L_{j'},$$
whence (1) follows. If the $L_j$ are temperate, then so also are the $K_{jt}$.

Conversely suppose (1) holds with $supp\ K_{jt} \subseteq \Gamma$. Let $A_{jt} = p_{jt}(D)\delta$ so $A \cdot K = \delta I$. Since the distributions with supports in $\Gamma$ form a commutative ring with respect to convolution it makes sense to take the determinant. From the Binet-Cauchy formula we obtain

$$\delta = \det(A \cdot K) = \sum'_{|J| = p} (\det A_{jt}) \cdot (\det K_{jt})$$

where $A_{jt}$ is the matrix whose $k$th column is the $j_k$th column of $A$ and $K_{jt}$ is the matrix whose $k$th row is the $j_k$th row of $K$. Since $A_{jt} = p_{jt}(D)(\delta I)$ we see that $\det A_{jt} = p_{jt}(D)\delta$. If we set $L_j = detK_j$ then (2) follows and $suppL_j \subseteq \Gamma$. If $K$ is temperate then the $L_j$ are temperate (see Appendix). Note it is not difficult to see if we start with $K$ and set $L_j = det K_j$ then the construction at the beginning of the proof yields the original $K$.

The lemma is now proved and moreover Theorem 1 follows from the $p = 1$ case which is considered in [8], [9]. The proof of the lemma is quite standard. The argument for example is similar to the argument in the $p = q$ case given in [1, Lemma 3.2]. The sufficiency of (2) in the $p < q$ case is the same as the argument in [11, Theorem 4.1]. We gave the argument, however, because prior to proceeding to the proof of Theorem 2 we will use the notation and proof of the lemma to solve an overdetermined Cauchy problem for a half-space when compatibility conditions are satisfied. Let $P'(z)$ denote the transpose of the matrix $P(z)$.

**Theorem 3.** Let $\Gamma$ be a closed convex salient cone and $p < q$. Assume (1) holds with $supp\ K \subseteq \Gamma$. Let $\eta \in \Gamma_{p}'$ and let $H$ be the closed half-space $\{x \in \mathbb{R}^n | \langle x, \eta \rangle > 0\}$. If $w \in (\mathbb{R})^p$ and if $supp(P'(D)w) \subseteq H$ then there exists a unique $u \in (\mathbb{R})^p$ such that

$$supp\ u \subseteq H, \quad P'(D)u = P'(D)w.$$ 

Moreover, if $w \in \mathbb{S}^p$ then $u \in \mathbb{S}^p$.

We prove uniqueness first. Suppose $u \in (\mathbb{R})_{H}^p$ and let $v = K' \cdot P'(D)u$. Since $K_{jk} \in \mathbb{R}_{\Gamma}'$ we have

$$v_k = \sum_j K_{jk} \cdot \sum_h p_{hj}(D)u_h$$

$$= \sum_{j,h} p_{hj}(D)K_{jk} \cdot u_h = u_k.$$ 

Thus $u = K' \cdot P'(D)u$ for any $u \in (\mathbb{R})_{H}^p$ which gives the uniqueness.

For existence we define $u \in (\mathbb{R})_{H}^p$ by $u = K' \cdot P'(D)w$. Note if $w$ is smooth, then so is $u$ which gives the last part. To see that $u$ is a solution, since we have no control over $supp\ w$ some care is required in commuting convolutions and differentiations. By the proof of the lemma we have $L_j \in \mathbb{R}_{\Gamma}'$ such that

$$K_{jk} = \sum_{h=1}^{p} \sum' Q_{jh}^{bh}(D)\ L_j$$

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where the inner sum is over $|J| = p$ with $j_h = j$. Then, since $\text{supp} \ L_j \subseteq \Gamma$,

$$u_k = \sum_j K_{jk} \cdot P_j(D) w_i$$

$$= \sum_{|J| = p} \sum_h Q_j^{kh}(D) L_j \cdot \sum_i P_{ih}(D) w_i$$

$$= \sum_{|J| = p} L_j \cdot \sum_{i, h} Q_j^{kh}(D) P_{ih}(D) w_i.$$

Now

$$Q_j(D) w_k = \sum_{i, h} Q_j^{kh}(D) P_{ih}(D) w_i$$

implies $Q_j(D) w_k$ has support in $H$ for each $J$ and each $k$. From the above computation we have

$$u_k = \sum_{|J| = p} L_j \cdot Q_j(D) w_k$$

and therefore

$$\sum_k P_{kl}(D) u_k = \sum_{|J| = p} L_j \cdot Q_j(D) \sum_k P_{kl}(D) w_k$$

$$= \sum_{|J| = p} Q_j(D) L_j \cdot \sum_k P_{kl}(D) w_k$$

$$= \sum_k P_{kl}(D) w_k$$

where the first equality follows from the fact that $Q_j(D) w_k$ has support in $H$ and the second from the fact that $P'(D) w$ has support in $H$.

**Proof of Theorem 2.** By the lemma we may assume $p = 1$. Thus $P(z) = (P_1(z), \ldots, P_q(z))$. Suppose first that $P_1(z), \ldots, P_q(z)$ have no common zero in $U + iR^n$ where $U$ is a convex open subset of $\Gamma_0^+$ such that $tU \subseteq U$ if $t > 1$ and $\Gamma_0^+$ is the union of $tU$ for $t > 0$. Locally in $U + iR^n$ we can find holomorphic functions $F_j$ such that $\sum P_j(z) F_j(z) = 1$. By Cartan's Theorem B [6, Theorem 7.4.3] these local solutions may be modified to fit together to give global holomorphic functions $F_j$ (here we use the convexity of $U$). Moreover by [8, Theorem 1] we may choose the holomorphic functions $F_j$ so that

$$|F_j(z)| < C(1 + |z|)^N d(\xi)^{-m}, \quad z \in U + iR^n,$$

for some constants $C$, $N$ and $m$. Here $\xi$ is the real part of $z$ and $d(\xi)$ is the minimum of 1 and the distance from $\xi$ to the boundary of $U$. By [10, Proposition 6, p. 306] $F_j$ is the Laplace transform of a distribution $K_j$. Then $\sum P_j(D) K_j = \delta$ and it remains to locate the support of $K_j$. That $\text{supp} \ K_j$ is contained in $\Gamma$ follows directly by estimating

$$\langle K_j, \phi \rangle = (2\pi)^{-n} \int F_j(\xi + i\eta) \bar{\phi}(i\xi - \eta) d\eta$$

where $\phi \in \mathcal{S}$ has support in a compact convex set disjoint from $\Gamma$ and $\bar{\phi}$ is the Fourier transform of $\phi$. The integral is independent of $\xi \in \Gamma_0^+$ and we simply
separate \( \Gamma \) and \( \text{supp } \hat{\phi} \) by a hyperplane with normal \( \xi \in \Gamma_0^+ \) and let \( |\xi| \to \infty \). Alternately \( \text{supp } K_j \) is contained in \( \Gamma \) by [10, Remark 1, p. 310].

For the converse we modify the argument in [1, Theorem 3.5]. Assume there exist \( K_j \in \mathcal{D}_r'^{+} \) such that \( \Sigma P_j(D)K_j = \delta \). Choose \( \phi \in \mathcal{S} \) with \( \phi(x) = 1 \) if \( |x| < 1 \) and \( \phi(x) = 0 \) if \( |x| > 2 \). Then \( \Sigma P_j(D)(\phi K_j) = \delta + g \) where \( g \in \mathcal{D}_r' \) and \( \text{supp } g \subseteq \{ x \in \Gamma | 1 < |x| < 2 \} \). Let \( G_j \) be the Laplace transform of \( \phi K_j \) and let \( G \) be the Laplace transform of \( g \). Then \( G \) and the \( G_j \) are entire functions and \( \Sigma P_j(z)G_j(z) = 1 + G(z) \). By the Paley-Wiener theorem [2, p. 211]

\[
|G(z)| < C(1 + |z|)^N e^{h(\xi)}
\]

where \( z = \xi + i\eta \) and where \( h(\xi) = \sup \{-\langle \xi, x \rangle | x \in \Gamma, 1 < |x| < 2 \} \). If \( \xi \in \Gamma_0^+ \) then \( \langle \xi, x \rangle > 0 \) for each \( x \in \Gamma \), \( x \neq 0 \) and hence \( h(\xi) = -\text{dist}(\xi, \partial \Gamma_0^+) \). Here \( \text{dist}(\xi, \partial \Gamma_0^+) = \inf \{ \langle \xi, x \rangle | x \in \Gamma, |x| = 1 \} \) is easily seen to be the distance from \( \xi \) to the boundary of \( \Gamma_0^+ \). At any common zero of the \( P_j \) we have \( G(z) = -1 \). Thus for some constants \( C \) and \( N \) we have

\[
\text{dist}(\xi, \partial \Gamma_0^+) < C + N \log(1 + |z|)
\]

if \( \xi \in \Gamma_0^+, z = \xi + i\eta \) and \( P_j(z) = 0, j = 1, \ldots, q \).

Suppose now \( \Gamma \) is semialgebraic. First note \( \Gamma_0^+ \) is the complement of the projection on the first \( n \) coordinates of the set of \( (\xi, x) \) such that \( \xi \in \mathbb{R}^n, x \in \Gamma, x \neq 0, \langle \xi, x \rangle < 0 \) and hence is semialgebraic by the Seidenberg-Tarski theorem. It follows that the set of \( (\mu, \xi, x) \) such that \( \xi \in \mathbb{R}^n, x \in \Gamma, |x| = 1, \mu > \langle \xi, x \rangle \) is semialgebraic and hence by the Seidenberg-Tarski theorem the set \( M \) of \( (\mu, \xi) \) such that \( \xi \in \Gamma_0^+ \) and \( \mu > \text{dist}(\xi, \partial \Gamma_0^+) \) is semialgebraic. An application of the Seidenberg-Tarski theorem shows that the closure and interior of a semialgebraic set is semialgebraic. Thus \( \partial M \cap (R \times \Gamma_0^+) = \{ (\mu, \xi) | \xi \in \Gamma_0^+, \mu = \text{dist}(\xi, \partial \Gamma_0^+) \} \) is semialgebraic. This property of the distance function, that the graph is semialgebraic, is known in other cases as well but is particularly simple to prove in our case because we have a nice formula for the distance to the boundary of a convex cone. It now follows that the set \( L_0 \) of \( (\mu, \tau, \xi, \eta) \) such that \( \xi \in \Gamma_0^+, \mu = \text{dist}(\xi, \partial \Gamma_0^+), \tau > |\xi + i\eta|, P_j(\xi + i\eta) = 0, j = 1, \ldots, q \), is semialgebraic. Again by the Seidenberg-Tarski theorem the projection \( L \) on the first two coordinates is semialgebraic. By (4) if \( (\mu, \tau) \in L \) then \( \mu < C + N \log(1 + \tau) \). By [5, Lemma 2.1, p. 276] it follows that there is a constant \( C_1 \) such that \( \mu < C_1 \) if \( (\mu, \tau) \in L \). Now let \( U = \{ \xi \in \Gamma_0^{|\text{dist}(\xi, \partial \Gamma_0^+)} > C_1 \} \).

Appendix. We now show \( S_\Gamma' \) is a subring of \( \mathcal{D}_r'^{+} \). Suppose \( f, g \in S_\Gamma' \). Then

\[
e^{-\langle \xi, \cdot \rangle}(f \ast g) = (e^{-\langle \xi, \cdot \rangle}f) \ast (e^{-\langle \xi, \cdot \rangle}g).
\]

For each \( \xi \in \Gamma_0^+ \) the factors on the right are in \( S' \) and therefore by [10, Corollary, p. 302] are in \( \Theta_\Gamma' \). It follows that \( e^{-\langle \xi, \cdot \rangle}(f \ast g) \) is in \( \Theta_\Gamma' \). Taking Fourier transforms we obtain

\[
(e^{-\langle \xi, \cdot \rangle}(f \ast g))'((\eta)) = F(\xi + i\eta)G(\xi + i\eta)
\]

where \( F \) (respectively \( G \)) is the Fourier transform of \( f \) (respectively \( g \)). The product on the right is a holomorphic function in \( \Gamma_0^+ + i\mathbb{R}^n \) and by [7, Theorem 1] is the Laplace transform of a distribution \( u \in S_\Gamma' \). Obviously \( u = f \ast g \).
References


Department of Mathematics, Oregon State University, Corvallis, Oregon 97331