SYSTEMS OF PARTIAL DIFFERENTIAL OPERATORS
WITH FUNDAMENTAL SOLUTIONS
SUPPORTED BY A CONE

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ABSTRACT. Necessary and sufficient conditions are given for a system of partial differential operators to have a fundamental solution supported by a convex salient cone. As a simple application an overdetermined Cauchy problem is solved.

If \( A \) is a subset of \( \mathbb{R}^n \) and \( \mathcal{F} \) is a space of distributions on \( \mathbb{R}^n \) we denote by \( \mathcal{F}_A \) the space of distributions in \( \mathcal{F} \) which have supports contained in \( A \). We denote by \( \mathcal{D}' \) the space of all distributions on \( \mathbb{R}^n \), by \( \mathcal{S}' \) the space of temperate distributions, and by \( \mathcal{S} \) the space of infinitely differentiable functions on \( \mathbb{R}^n \). If \( \Gamma \) is a closed convex cone in \( \mathbb{R}^n \) with vertex at the origin, we denote by \( \Gamma^+ \) the dual cone defined by \( \{ \xi \in \mathbb{R}^n | \langle \xi, x \rangle > 0, x \in \Gamma \} \). Then \( \Gamma^{++} = \Gamma \). The interior \( \Gamma_0^+ \) of \( \Gamma^+ \) is nonempty if and only if \( \Gamma \) is salient, i.e. contains no subspace other than \( \{0\} \). If \( \Gamma \) is salient then \( \mathcal{D}'_\Gamma \) is a commutative ring relative to convolution. If \( H \) is a closed half-space with interior normal \( \eta \in \Gamma_0^+ \) then \( \mathcal{D}'_H \) is a \( \mathcal{D}'_\Gamma \)-module, and differentiation commutes with convolution in the usual fashion. Finally we note \( \mathcal{S}'_\Gamma \) is a subring of \( \mathcal{D}'_\Gamma \). This fact is proved in the appendix below.

Let \( P(z) \) be a \( p \times q \) matrix over \( C[z_1, \ldots, z_n] \) and denote by \( P(D) \) the system of partial differential operators obtained by replacing \( z_j \) in \( P(z) \) by \( \partial / \partial x_j \). If \( p < q \) then a fundamental solution for \( P(D) \) is a \( q \times p \) matrix \( K \) over \( \mathcal{D}' \) such that

\[
P(D)K = \delta I
\]

where \( I \) is the \( p \times p \) identity matrix and \( \delta \) is the Dirac measure at \( 0 \). In case \( p = q \) then \( P(D) \) has a fundamental solution with support in the closed convex salient cone \( \Gamma \) if and only if \( P(D) \) is hyperbolic with respect to each direction in \( \Gamma_0^+ \), [1]. In case \( p = q = 1 \) then \( P(D) \) has a temperate fundamental solution with support in the closed convex salient cone \( \Gamma \) if and only if \( P(z) \neq 0 \) for each \( z \) in \( \Gamma_0^+ + i\mathbb{R}^n \). This fact may be proved by means of an elementary inequality for polynomials, as is done in the introduction to [9]. The temperate case with \( p = 1, q > 1 \) is also considered in ([8], [9]) and may easily be generalized as is done below. In this note we will give a sufficient, and in case \( \Gamma \) is semialgebraic, necessary condition for \( P(D) \) in the case \( p < q \) to have a fundamental solution \( K \) with support in the closed convex salient cone \( \Gamma \). Our methods do not apply in the nonsalient case. The scalar case \( p = q = 1 \) with \( \Gamma \) nonsalient has been considered by A. Enqvist in [3] and in the temperate case in [4]. We will prove the following two theorems.

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Theorem 1. Let $\Gamma$ be a closed convex salient cone and $p < q$. Then $P(D)$ admits a temperate fundamental solution $K$ with support in $\Gamma$ if and only if for each $z$ in the tube $\Gamma_0^+ + i\mathbb{R}^n$ the matrix $P(z)$ has rank $p$.

Theorem 2. Let $\Gamma$ be a closed convex salient cone and $p < q$. Then $P(D)$ admits a fundamental solution $K$ with support in $\Gamma$ if and only if for each $z$ in the tube $U + i\mathbb{R}^n$ the matrix $P(z)$ has rank $p$.

In the case $p = q$ we may dispense with the hypothesis that $\Gamma$ is semialgebraic. There are at least two ways to do this. If $P(D)$ has a fundamental solution with support in $\Gamma$ then the determinant $\det P(D)$ is hyperbolic with respect to each direction in $\Gamma_0^+$. From the theory of scalar hyperbolic operators [5] it follows that $\det P(D)$ is hyperbolic with respect to each direction in an open convex semialgebraic cone which contains $\Gamma_0^+$. By the lemma below we then obtain a closed convex semialgebraic cone $\Gamma' \subseteq \Gamma$ such that $P(D)$ has a fundamental solution with support in $\Gamma'$. Alternately, if $\Gamma$ is not assumed semialgebraic a modification of the proof of necessity produces an open set $U$ with the required properties other than convexity. In the case $p = q$, S. Bochner's theorem on tubes [6, Theorem 2.5.10] then shows we may replace $U$ by its convex hull.

We first reduce the $p \times q$ system to a $1 \times N$ system, $N = (q)_p$. The notation $|J| = p$ will mean that $J = (j_1, \ldots, j_p)$ where the $j_k$ are integers and $1 < j_k < q$ for each $k$. For each such $J$ let $P^J(z)$ be the $p \times p$ matrix whose $k$th column is the $j_k$th column of $P(z)$ and let $Q_j(z)$ be the determinant of $P^J(z)$.

Lemma. Let $\Gamma$ be a closed convex salient cone and $p < q$. Then $P(D)$ admits a fundamental solution (respectively, a temperate fundamental solution) with support in $\Gamma$ if and only if there exist distributions (respectively, temperate distributions) $L_j$, $|J| = p$, with supports in $\Gamma$ such that

$$\sum_{|J| = p} Q_j(D)L_j = \delta.$$  \hspace{1cm} (2)

Here the prime over the summation symbol indicates that we sum only over $p$-indices $J = (j_1, \ldots, j_p)$ with $1 < j_1 < \cdots < j_p < q$. For the proof, suppose first that (2) holds with supp $L_j \subseteq \Gamma$. Let $Q^h_i(z)$ be the $(i, k)$-cofactor of $P^J(z)$, that is $(-1)^{+\kappa}$ times the determinant of the matrix obtained from $P^J(z)$ by removing the $i$th row and the $k$th column. Then

$$\sum_{h=1}^{p} P_{ih}(z)Q^h_i(z) = \begin{cases} Q_i(z) & \text{if } l = i, \\ 0 & \text{if } l \neq i, \end{cases}$$

where $J = (j_1, \ldots, j_p)$. If we set

$$K_{jl} = \sum_{h=1}^{p} \sum Q^h_i(D)L_j, \quad 1 < j < q, 1 < l < p,$$

where the inner sum is over $|J| = p$ such that $j_h = j$, then supp $K_{jl} \subseteq \Gamma$ and

$$\sum_{j=1}^{q} P_j(D)K_{jl} = \sum_{|J| = p} \sum_{h=1}^{p} P_{ih}(D)Q^h_i(D)L_j,$$
whence (1) follows. If the $L_j$ are temperate, then so also are the $K_{ji}$.

Conversely suppose (1) holds with $\text{supp } K_{ji} \subseteq \Gamma$. Let $A_{ji} = P_{ji}(D)\delta$ so $A \cdot K = \delta I$. Since the distributions with supports in $\Gamma$ form a commutative ring with respect to convolution it makes sense to take the determinant. From the Binet-Cauchy formula we obtain

$$\delta = \det(A \cdot K) = \sum'_{|J|=p} (\det A_{ji}) \cdot (\det K_{ji})$$

where $A_{ji}$ is the matrix whose $k$th column is the $j_k$th column of $A$ and $K_{ji}$ is the matrix whose $k$th row is the $j_k$th row of $K$. Since $A_{ji} = P_{ij}(D)(\delta I)$ we see that $\det A_{ji} = Q_{ij}(D)\delta$. If we set $L_{ij} = \det K_{ji}$ then (2) follows and $\text{supp } L_{ij} \subseteq \Gamma$. If $K$ is temperate then the $L_{ij}$ are temperate (see Appendix). Note it is not difficult to see if we start with $K$ and set $L_{ij} = \det K_{ji}$ then the construction at the beginning of the proof yields the original $K$.

The lemma is now proved and moreover Theorem 1 follows from the $p = 1$ case which is considered in [8], [9]. The proof of the lemma is quite standard. The argument for example is similar to the argument in the $p = q$ case given in [1, Lemma 3.2]. The sufficiency of (2) in the $p < q$ case is the same as the argument in [11, Theorem 4.1]. We gave the argument, however, because prior to proceeding to the proof of Theorem 2 we will use the notation and proof of the lemma to solve an overdetermined Cauchy problem for a half-space when compatibility conditions are satisfied. Let $P'(z)$ denote the transpose of the matrix $P(z)$.

**Theorem 3.** Let $\Gamma$ be a closed convex salient cone and $p < q$. Assume (1) holds with $\text{supp } K \subseteq \Gamma$. Let $\eta \in \Gamma'_{\nu}$ and let $H$ be the closed half-space $\{x \in \mathbb{R}^n | \langle x, \eta \rangle > 0 \}$. If $w \in (\otimes)'^\rho$ and if $\text{supp}(P'(D)w) \subseteq H$ then there exists a unique $u \in (\otimes)'^\rho$ such that

$$\text{supp } u \subseteq H, \quad P'(D)u = P'(D)w.$$  

Moreover, if $w \in \mathcal{E}^\rho$ then $u \in \mathcal{E}^\rho$.

We prove uniqueness first. Suppose $u \in (\otimes)'^\rho$ and let $v = K' \cdot P'(D)u$. Since $K_{jk} \in \otimes'_{\Gamma'}$ we have

$$v_k = \sum_j K_{jk} \cdot \sum_h P_{jh}(D)u_h$$

$$= \sum_{j,h} P_{jh}(D)K_{jk} \cdot u_h = u_k.$$  

Thus $u = K' \cdot P'(D)u$ for any $u \in (\otimes)'_{\nu}$ which gives the uniqueness.

For existence we define $u \in (\otimes)'^\rho$ by $u = K' \cdot P'(D)w$. Note if $w$ is smooth, then so is $u$ which gives the last part. To see that $u$ is a solution, since we have no control over supp $w$ some care is required in commuting convolutions and differentiations. By the proof of the lemma we have $L_{ij} \in \otimes'_{\Gamma'}$ such that

$$K_{jk} = \sum_{h=1}^p \sum'_{Q_{kj}^h(D)} L_{ij}$$
where the inner sum is over $|J| = p$ with $j_h = j$. Then, since $\text{supp } L_J \subseteq \Gamma$,

$$u_k = \sum_j K_{jk} \cdot P_j(D)w_i$$

$$= \sum'_{|J|=p} \frac{1}{h} \sum_i Q_{j,h_k}^k(D) L_J \cdot \sum_h P_{ij_h}(D)w_i$$

$$= \sum'_{|J|=p} L_J \cdot \sum_{i,h} Q_{j,h_k}^k(D) P_{ij_h}(D)w_i.$$ 

Now

$$Q_j(D)w_k = \sum_{i,h} Q_{j,h_k}^k(D) P_{ij_h}(D)w_i$$

implies $Q_j(D)w_k$ has support in $H$ for each $J$ and each $k$. From the above computation we have

$$u_k = \sum'_{|J|=p} L_J \cdot Q_j(D)w_k$$

and therefore

$$\sum_k P_{kl}(D)u_k = \sum'_{|J|=p} L_J \cdot Q_j(D) \sum_k P_{kl}(D)w_k$$

$$= \sum'_{|J|=p} Q_j(D) L_J \cdot \sum_k P_{kl}(D)w_k$$

$$= \sum_k P_{kl}(D)w_k$$

where the first equality follows from the fact that $Q_j(D)w_k$ has support in $H$ and the second from the fact that $P'(D)w$ has support in $H$.

**Proof of Theorem 2.** By the lemma we may assume $p = 1$. Thus $P(z) = (P_1(z), \ldots, P_q(z))$. Suppose first that $P_1(z), \ldots, P_q(z)$ have no common zero in $U + iR^n$ where $U$ is a convex open subset of $\Gamma^+$ such that $tU \subseteq U$ if $t > 1$ and $\Gamma^+$ is the union of $tU$ for $t > 0$. Locally in $U + iR^n$ we can find holomorphic functions $F_j$ such that $\sum P_j(z)F_j(z) = 1$. By Cartan's Theorem B [6, Theorem 7.4.3] these local solutions may be modified to fit together to give global holomorphic functions $F_j$ (here we use the convexity of $U$). Moreover by [8, Theorem 1] we may choose the holomorphic functions $F_j$ so that

$$|F_j(z)| < C(1 + |z|)^{m}d(\xi)^{-m}, \quad z \in U + iR^n,$$

for some constants $C, N$ and $m$. Here $\xi$ is the real part of $z$ and $d(\xi)$ is the minimum of 1 and the distance from $\xi$ to the boundary of $U$. By [10, Proposition 6, p. 306] $F_j$ is the Laplace transform of a distribution $K_j$. Then $\sum P_j(D)K_j = \delta$ and it remains to locate the support of $K_j$. That supp $K_j$ is contained in $\Gamma$ follows directly by estimating

$$\langle K_j, \phi \rangle = (2\pi)^{-n} \int F_j(\xi + \eta)\tilde{\phi}(i\xi - \eta)d\eta$$

where $\phi \in \mathcal{E}$ has support in a compact convex set disjoint from $\Gamma$ and $\tilde{\phi}$ is the Fourier transform of $\phi$. The integral is independent of $\xi \in \Gamma^+$ and we simply
separate $\Gamma$ and $\text{supp} \, \hat{\phi}$ by a hyperplane with normal $\xi \in \Gamma_0^+$ and let $|\xi| \to \infty$. Alternately $\text{supp} \, K_j$ is contained in $\Gamma$ by [10, Remark 1, p. 310].

For the converse we modify the argument in [1, Theorem 3.5]. Assume there exist $K_j \in \mathcal{D}_\mathcal{C}^+$ such that $\sum P_j(D)K_j = \delta$. Choose $\phi \in \mathcal{S}$ with $\phi(x) = 1$ if $|x| < 1$ and $\phi(x) = 0$ if $|x| > 2$. Then $\sum P_j(D)(\phi K_j) = \delta + g$ where $g \in L^\infty$ and $\text{supp} \, g \subseteq \{x \in \mathbb{R}^n | 1 < |x| < 2\}$. Let $G_j$ be the Laplace transform of $\phi K_j$ and let $G$ be the Laplace transform of $g$. Then $G$ and the $G_j$ are entire functions and $\sum P_j(z)G_j(z) = 1 + G(z)$. By the Paley-Wiener theorem [2, p. 211]

$$|G(z)| < C(1 + |z|)^N e^{h(-z)}$$

where $z = \xi + i\eta$ and where $h(-\xi) = \sup\{\langle \xi, x \rangle | x \in \mathcal{E}, 1 < |x| < 2\}$. If $\xi \in \Gamma_0^+$ then $\langle \xi, x \rangle > 0$ for each $x \in \mathcal{E}$, $x \neq 0$ and hence $h(-\xi) = -\text{dist}(\xi, \partial \mathcal{E}^+)$. Here $\text{dist}(\xi, \partial \mathcal{E}^+) = \inf\{\langle \xi, x \rangle | x \in \mathcal{E}, |x| = 1\}$ is easily seen to be the distance from $\xi$ to the boundary of $\mathcal{E}^+$. At any common zero of the $P_j$ we have $G(z) = -1$. Thus for some constants $C$ and $N$ we have

$$\text{dist}(\xi, \partial \mathcal{E}^+) < C + N \log(1 + |z|)$$

if $\xi \in \Gamma_0^+$, $z = \xi + i\eta$ and $P_j(z) = 0, j = 1, \ldots, q$.

Suppose now $\Gamma$ is semialgebraic. First note $\Gamma_0^+$ is the complement of the projection on the first $n$ coordinates of the set of $(\xi, x)$ such that $\xi \in \mathbb{R}^n$, $x \in \mathcal{E}$, $x \neq 0$, $\langle \xi, x \rangle < 0$ and hence is semialgebraic by the Seidenberg-Tarski theorem. It follows that the set of $(\mu, \xi, x)$ such that $\xi \in \Gamma_0^+$, $x \in \mathcal{E}, |x| = 1$, $\mu > \langle \xi, x \rangle$ is semialgebraic and hence by the Seidenberg-Tarski theorem the set $M$ of $(\mu, \xi)$ such that $\xi \in \Gamma_0^+$ and $\mu > \text{dist}(\xi, \partial \mathcal{E}^+)$ is semialgebraic. An application of the Seidenberg-Tarski theorem shows that the closure and interior of a semialgebraic set is semialgebraic. Thus $\delta M \cap (R \times \Gamma_0^+) = \{(\mu, \xi) | \xi \in \Gamma_0^+, \mu = \text{dist}(\xi, \partial \mathcal{E}^+)\}$ is semialgebraic. This property of the distance function, that the graph is semialgebraic, is known in other cases as well but is particularly simple to prove in our case because we have a nice formula for the distance to the boundary of a convex cone. It now follows that the set $L_0$ of $(\mu, \tau, \xi, \eta)$ such that $\xi \in \Gamma_0^+, \mu = \text{dist}(\xi, \partial \mathcal{E}^+)$, $\tau > |\xi + i\eta|$, $P_j(\xi + i\eta) = 0, j = 1, \ldots, q$, is semialgebraic. Again by the Seidenberg-Tarski theorem the projection $L$ on the first two coordinates is semialgebraic. By (4) if $(\mu, \tau) \in L$ then $\mu < C + N \log(1 + \tau)$. By [5, Lemma 2.1, p. 276] it follows that there is a constant $C_1$ such that $\mu < C_1$ if $(\mu, \tau) \in L$. Now let $U = \{\xi \in \Gamma_0^+ | \text{dist}(\xi, \partial \mathcal{E}^+) > C_1\}$.

**Appendix.** We now show $S'_\mathcal{C}$ is a subring of $\mathcal{D}_\mathcal{C}$. Suppose $f, g \in S'_\mathcal{C}$. Then

$$e^{-\langle \xi, \cdot \rangle}(f * g) = (e^{-\langle \xi, \cdot \rangle}f) \ast (e^{-\langle \xi, \cdot \rangle}g).$$

For each $\xi \in \Gamma_0^+$ the factors on the right are in $S'$ and therefore by [10, Corollary, p. 302] are in $\mathcal{D}_\mathcal{C}$. It follows that $e^{-\langle \xi, \cdot \rangle}(f * g)$ is in $\mathcal{D}_\mathcal{C}$. Taking Fourier transforms we obtain

$$(e^{-\langle \xi, \cdot \rangle}(f * g))(\eta) = F(\xi + i\eta)G(\xi + i\eta)$$

where $F$ (respectively $G$) is the Fourier transform of $f$ (respectively $g$). The product on the right is a holomorphic function in $\Gamma_0^+ + i\mathbb{R}^n$ and by [7, Theorem 1] is the Laplace transform of a distribution $u \in S'_\mathcal{C}$. Obviously $u = f * g$. 

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