INTERPOLATING FUNCTIONS ASSOCIATED WITH SECOND-ORDER DIFFERENTIAL EQUATIONS

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Abstract. Functions are exhibited which interpolate the magnitude of a solution \( y \) of a linear, homogeneous, second-order differential equation at its critical points, \(|y'|\) at the zeros of \( y \), and \(|\int_{x_0}^{x} y(t)h(t)\,dt|\) at the zeros of \( y \). Except for a normalization condition, the interpolating functions are independent of the specific solution \( y \). A theorem similar in its conclusions to the Sonin-Pólya-Butlewski theorem is presented and examples are given.

1. Introduction. Suppose \( y_1 \) and \( y_2 \) form a fundamental system for the differential equation

\[
(p(x)y')' + q(x)y = 0, \quad a < x < b,
\]

where we assume throughout that \( p \) and \( q \) are continuous and \( p > 0 \) on \((a, b)\). Let \( y = Ay_1 + By_2 \) be an arbitrary nontrivial solution of (1), normalized so that \( A^2 + B^2 = 1 \). We will exhibit a function, independent of \( A \) and \( B \), which interpolates \(|y|\) at the critical points of \( y \), and others, also independent of \( A \) and \( B \), which interpolate \(|y'|\) and \(|\int_{x_0}^{x} y(t)h(t)\,dt|\) at the zeros of \( y \).

2. Preliminary lemmas.

Lemma 1. Suppose \( f_1, f_2, g_1 \) and \( g_2 \) are real numbers, with \( f_1^2 + f_2^2 > 0 \). Let

\[
f = Af_1 + Bf_2, \quad g = Ag_1 + Bg_2,
\]

with

\[
A^2 + B^2 = 1.
\]

Then

\[
|g| = \left| f_1 g_2 - f_2 g_1 \right| / \left( f_1^2 + f_2^2 \right)^{1/2}, \text{ if } f = 0.
\]

Proof. The conclusion follows from the identity

\[
(A^2 + B^2)(f_1 g_2 - f_2 g_1)^2 = (f_2 g - f_1 g)^2 + (f_1 g - f_2 g_1)^2,
\]

which can be derived by solving (2) for \( A \) and \( B \), squaring and adding, and then multiplying both sides of the resulting equation by \((f_1 g_2 - f_2 g_1)^2\). Although this seemingly requires the assumption that \( f_1 g_2 - f_2 g_1 \neq 0 \), it is easily verified that (4) holds for all values of the quantities appearing in it. Invoking (3) yields (5).

In the next two lemmas \( y_1 \) and \( y_2 \) need not be solutions of (1).

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Lemma 2. Let $y_1$ and $y_2$ be differentiable and suppose their Wronskian, $W = y_1y_2' - y_2y_1'$, has no zeros in $(a, b)$. Let

$$y = Ay_1 + By_2 \quad (A^2 + B^2 = 1).$$

Then

$$|y(x)| = |W(x)| \left[ (y_1'(x))^2 + (y_2'(x))^2 \right]^{1/2} \quad \text{if } x \in (a, b) \text{ and } y'(x) = 0,$$

and

$$|y'(x)| = |W(x)| \left[ y_1^2(x) + y_2^2(x) \right]^{1/2} \quad \text{if } x \in (a, b) \text{ and } y(x) = 0. \quad (8)$$

**Proof.** To obtain (7), take $f_i = y_i'(x)$ and $g_i = y_i(x)$ ($i = 1, 2$) in Lemma 1. To obtain (8), take $f_i = y_i(x)$ and $g_i = y_i'(x)$ in Lemma 1. The nonvanishing of $W$ guarantees that the denominators in (7) and (8) are nonzero.

Lemma 3. Suppose $y_1h$ and $y_2h$ are locally integrable and $y_1^2 + y_2^2 > 0$ on $(a, b)$. Define

$$s(x) = \int_{x_0}^{x} [y_1(t)y_2(x) - y_2(t)y_1(x)] h(t) \, dt, \quad a < x_0 < x < b, \quad (9)$$

and let $y$ satisfy (6). Then

$$\int_{x_0}^{x} y(t)h(t) \, dt = \frac{|s(x)|}{\left[ y_1^2(x) + y_2^2(x) \right]^{1/2}} \quad \text{if } x \in (a, b) \text{ and } y(x) = 0. \quad (10)$$

The conclusion also holds with $x_0 = a$ if $\int_{a}^{x} y_i(t)h(t) \, dt$ exists ($i = 1, 2$), or with $x_0 = b$ if $\int_{b}^{x} y_i(t)h(t) \, dt$ exists ($i = 1, 2$).

**Proof.** Take $f_i = y_i(x)$ and $g_i = \int_{x_0}^{x} y_i(t)h(t) \, dt$ ($i = 1, 2$) in Lemma 1.

3. **Main results.** If $y_1$ and $y_2$ are linearly independent solutions of (1), then

$$y_1y_2' - y_2y_1' = k/p \quad (k = \text{constant} \neq 0). \quad (11)$$

This and Lemma 2 yield the following theorem.

**Theorem 1.** Let $y_1$ and $y_2$ be solutions of (1) satisfying (11), and suppose $y$ satisfies (6). Then the function

$$I_1 = |k|/p \left[ (y_1')^2 + (y_2')^2 \right]^{1/2} \quad (12)$$

interpolates $|y|$ at the critical points of $y$, and the function

$$I_2 = |k|/p \left( y_1^2 + y_2^2 \right)^{1/2}$$

interpolates $|y'|$ at the zeros of $y$.

This theorem can essentially be obtained by applying known transformations to (1), although the argument presented above is simpler and appears to require fewer assumptions. The substitutions of

$$t = \int_{x_0}^{x} \frac{du}{p(u)}, \quad Y(t) = y(x), \quad (13)$$
transform (1) into
\[ \frac{d^2Y}{dt^2} + Q(t)Y = 0, \]
where \( Q(t) = p(x)q(x) \). If \( y_1 \) and \( y_2 \) are as defined at the beginning of this section and \( Y_i(t) = y_i(t) \) (\( i = 1, 2 \)), let \( f(t) = Y_1(t) + Y_2(t) \). L. Lorch and P. Szegö [3] have shown that the substitutions
\[ z = \int f(t) \, \frac{du}{f(u)}, \quad u(z) = (f(t))^{1/2} Y(t) \]
transform (14) into \( \frac{d^2u}{dz^2} + k^2u = 0 \). Therefore, the general solution of (1) is
\[ y(x) = C[y_1^2(x) + y_2^2(x)]^{1/2} \sin\left(k \int f(t) \, \frac{du}{f(u)}\right), \]
with \( t \) related to \( x \) as in (13). (This formula was given by Borůvka [1, p. 43] for the case where \( p(x) = 1 \).) By differentiating (15) while recalling (13), and noting that \( y(x) = 0 \) if and only if \( \sin(k \int f(t) \, \frac{du}{f(u)}) = 0 \) (and so \( \cos(k \int f(t) \, \frac{du}{f(u)}) = \pm 1 \)), it can be shown that a constant multiple of \( I_2 \) interpolates \( |y'| \) at the zeros of \( y \).

Formally, the equation \( (u'/q)' + u/p = 0 \) has solutions \( u = py' \), where \( y \) satisfies (1). Applying the procedure of the last paragraph to this equation leads to a general formula for \( y' \) (also given by Borůvka for the case where \( p(x) = 1 \) [1, p. 43]), from which it can be shown that a constant multiple of \( I_1 \) interpolates \( |y| \) at the zeros of \( y' \). It would appear that this derivation of \( I_1 \) requires the additional assumption that \( p \) has no zeros on \( (a, b) \).

It should be observed that the use of the function \( y_1^2 + y_2^2 \) for obtaining qualitative properties of solutions of (1) occurs in many places in the literature; for examples, see [2, pp. 515–519], [4], [5] and [6].

**Theorem 2.** Suppose (1) has linearly independent solutions \( y_1 \) and \( y_2 \) such that the function \( F = q(y_1^2 + y_2^2)' \) does not change sign on \( (a, b) \). Let \( \{x_n\} \) be an increasing sequence of critical points of an arbitrary nontrivial solution \( y \) of (1). Then the sequence \( \{|y(x_n)|\} \) is nondecreasing if \( F > 0 \), or nonincreasing if \( F < 0 \). The monotonicity of \( \{|y(x_n)|\} \) is strict if \( F \) is not identically zero on any subinterval of \( (a, b) \).

**Proof.** Assume without loss of generality that \( y \) is normalized as in (6); then Theorem 1 implies that \( |y(x_n)| = I_1(x_n), \) with \( I_1 \) as in (12). Differentiating \( I_1 \) and noting that \( y_1 \) and \( y_2 \) satisfy (1) yields \( I_1' = pF I_1^3/2k^2 \). This implies the conclusion.

**Theorem 3.** Suppose \( y_1 \) and \( y_2 \) are solutions of (1) which satisfy (11), that \( a < x_0 < b \), and that \( s \) satisfies
\[ (p(x)s')' + q(x)s = kh(x), \quad s(x_0) = s'(x_0) = 0, \]
where \( h \) is continuous on \( (a, b) \). Then (10) holds, with \( y \) as in (6). This is also true with \( x_0 = a \) if \( \int_a y_i(t)h(t) \, dt \) exists (\( i = 1, 2 \)), or with \( x_0 = b \) if \( \int_b y_i(t)h(t) \, dt \) exists (\( i = 1, 2 \)).
Proof. By variation of parameters, \( s \) is as in (9), so Lemma 3 implies the conclusion.

4. Examples. Let (1) be Bessel's equation,

\[
(xy')' + \frac{1}{x}(x^2 - \nu^2)y = 0,
\]

with \( y_1 = J_\nu \) and \( y_2 = Y_\nu \), the Bessel functions of the first and second kinds, and let

\[
C_\nu = AJ_\nu + BY_\nu \quad (A^2 + B^2 = 1)
\]

be an arbitrary normalized cylinder function.

Example 1. Since the Wronskian of \( y_1 \) and \( y_2 \) in this case is

\[
W(x) = J_\nu(x)Y'_\nu(x) - J'_\nu(x)Y_\nu(x) = 2/\pi x
\]

[8, p. 76], Theorem 1 implies that

\[
|C_\nu(x)| = \frac{2}{\pi x[J_\nu(x)]^2 + [Y_\nu(x)]^2]^{1/2}} \quad \text{if } x > 0 \text{ and } C_\nu(x) = 0,
\]

and

\[
|C'_\nu(x)| = \frac{2}{\pi x[J_\nu(x)]^2 + [Y_\nu(x)]^2]^{1/2}} \quad \text{if } x > 0 \text{ and } C_\nu(x) = 0. \quad (18)
\]

It is perhaps worth noting that both interpolating functions are rational if \( \nu = k + 1/2 \), where \( k \) is an integer [8, p. 297]. Since \( x(J_\nu^2 + Y_\nu^2) \) increases if \( \nu > 1/2 \) and decreases if \( \nu < 1/2 \) [8, p. 466], (18) implies that if \( \{x_n\} \) is an increasing sequence of positive zeros of \( C_\nu \), then the sequence \( \{|C_\nu(x_n)|\} \) decreases if \( \nu > 1/2 \) and increases if \( \nu < 1/2 \). Since \( J_\nu^2 + Y_\nu^2 \) decreases for all \( \nu \) [8, p. 466], (18) also implies that the sequence \( \{x_n|C_\nu(x_n)|\} \) increases for all \( \nu \).

Example 2. If \( C_\nu \) is a cylinder function as in (16), then \( \int_0^x t^\mu C_\nu(t) \, dt \) exists if \( \mu + \nu > -1 \). From (11) and (17), the initial value problem associated with this integral according to Theorem 3 is

\[
(xs')' + \frac{1}{x}(x^2 - \nu^2)s = 2x^\mu/\pi, \quad s(0) = s'(0) = 0,
\]

which has the solution \( s = 2s_{\mu\nu}/\pi \), where \( s_{\mu\nu} \) is Lommel's function of the first kind [8, pp. 345–346]. Now (10) implies that

\[
\left| \int_0^x t^\mu C_\nu(t) \, dt \right| = \frac{2|s_{\mu\nu}(x)|/\pi}{[J_\nu^2(x) + Y_\nu^2(x)]^{1/2}} \quad \text{if } x > 0 \text{ and } C_\nu(x) = 0.
\]

Example 3. If \( \mu < 1/2 \) and \( C_\nu \) is as in (16), then \( \int_x^\infty t^\mu C_\nu(t) \, dt \) exists for \( x > 0 \). The initial value problem associated with this integral according to Theorem 3 is

\[
(xs')' + \frac{1}{x}(x^2 - \nu^2)s = 2x^\mu/\pi, \quad s(\infty) = s'(\infty) = 0,
\]

which has the solution \( s = 2s_{\mu\nu}/\pi \), where \( s_{\mu\nu} \) is Lommel's function of the
second kind [8, p. 347]. Therefore, Theorem 3 implies that
\[
\left| \int_{x}^{\infty} t^\alpha \mathcal{C}_\nu(t) \ dt \right| = \frac{2 |S_\nu(x)| / \pi}{\left[ J_\nu^2(x) + Y_\nu^2(x) \right]^{1/2}} \text{ if } x > 0 \text{ and } \mathcal{C}_\nu(x) = 0.
\]

**Example 4.** Theorem 2 and the fact that \( J_\nu^2 + Y_\nu^2 \) decreases imply that the successive maxima of \( |\mathcal{C}_\nu| \) on \((\nu, \infty)\) form a decreasing sequence. This is a known result [8, p. 488].

The conclusion of Theorem 2 is similar to that of the Sonin-Pólya-Butlewski theorem [7, p. 166], which says that if \( p > 0, q > 0, p \) and \( q \) are continuously differentiable on \((a, b)\), and \( pq \) is monotonic, then \( \{|y(x)|\} \) is monotonic in the opposite sense. By considering transformed versions of Bessel's equation and using known monotonicity properties of \( x(J_\nu^2 + Y_\nu^2) \) and \( (x^2 - \nu^2)^{1/2}(J_\nu^2 + Y_\nu^2) \) [8, p. 446], it is possible to obtain from Theorem 2 monotonicity properties of sequences of maxima of \( x^{1/2}|\mathcal{C}_\nu| \) and \( (x^2 - \nu^2)^{1/4}|\mathcal{C}_\nu| \); however, the differential equations in question also satisfy the hypotheses of the Sonin-Pólya-Butlewski theorem, and the results are not new. The following contrived example presents a result implied by Theorem 2 that cannot be obtained from the Sonin-Pólya-Butlewski theorem.

**Example 5.** If \( g \) is positive and continuously differentiable on \((0, \infty)\), then
\[
y_1(x) = x \cos \left( \int_{0}^{x} g(t) \ dt \right) \quad \text{and} \quad y_2(x) = x \sin \left( \int_{0}^{x} g(t) \ dt \right)
\]
form a fundamental system for
\[
\left( \frac{y'}{x^2 g} \right)' + \left( \frac{2}{x^3 g} + \frac{g'}{x^2 g^2} + \frac{g}{x} \right)y = 0, \quad x > 0.
\]
Clearly (19) is oscillatory if \( \int_{0}^{\infty} g(t) \ dt = \infty \), and since \( y_1^2(x) + y_2^2(x) = x^2 \) is increasing, it satisfies the hypotheses of Theorem 2 on \((0, \infty)\) if \( g' > 0 \); therefore, if these assumptions hold and \( \phi \) is any constant, then the absolute values of
\[
y(x) = x \cos \left( \phi + \int_{0}^{x} g(t) \ dt \right)
\]
at any increasing sequence of critical points of \( y \) form an increasing sequence. The Sonin-Pólya-Butlewski theorem does not imply this, since
\[
pq = \frac{1}{x^2 g} \left( \frac{2}{x^3 g} + \frac{g'}{x^2 g^2} + \frac{g}{x} \right)
\]
need not be monotonic; for example, let
\[
g(x) = \int_{0}^{x} (2 - \cos e^t) \ dt.
\]

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REFERENCES


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