ON BEURLING'S THEOREM FOR LOCALLY COMPACT GROUPS

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Abstract. Beurling's Theorem in spectral analysis of bounded functions on the real line is generalized to a class of semidirect products of locally compact abelian groups.

As an immediate consequence of Wiener's Tauberian Theorem one has, by duality (see [4, p. 181]), the following theorem on spectral analysis: Unless \( f \in L_\infty(\mathbb{R}) \) is zero almost everywhere, the \( \ast \)-closed subspace generated by the translates of \( f \) contains a function \( e^{i\lambda x} \) for some \( \lambda \in \mathbb{R} \). Beurling proved [1] much more about a smaller class of functions. His theorem is, essentially, the following:

**Beurling's Theorem.** Let \( f \) be a nonzero, bounded uniformly continuous function on \( \mathbb{R} \). Then there exists a real number \( \lambda \), such that the function \( e^{i\lambda x} \) belongs to the \( \ast \)-closure of some norm-bounded set of linear combinations of translates of \( f \).

This result was generalized to locally compact abelian groups [2], [3].

The purpose of this note is to generalize Beurling's Theorem to a class of semidirect products of locally compact abelian groups.

For the group \( G^* \) of the linear transformations on the real line of the form \( ax + b, a > 0, b \in \mathbb{R} \), the following result was announced in [5]: Every proper closed two-sided ideal of \( L_1(G^*) \) is contained in a maximal modular two-sided ideal. Our result implies, by duality, that every proper closed two-sided ideal of \( L_1(G^*) \) is contained, actually, in the kernel of a one-dimensional representation of \( G^* \) and that all closed two-sided maximal ideals of \( L_1(G^*) \) are of this type.

Let \( G = N \rtimes H \) denote the semidirect product of the groups \( N \) and \( H \) and let \( h \to \tau_h \) be the homomorphism which carries \( H \) onto a group of automorphisms of \( N \).

For a locally compact abelian group \( G \), let \( \hat{G} \) denote the character group of \( G \) and \( 1_G \) the identity element of \( G \).

We prove the following theorem:

**Theorem.** Let \( G = N \rtimes H \) where \( N \) and \( H \) are locally compact abelian groups. Suppose that for every \( \chi \in \hat{N} \), there exists a sequence \( \{ h_k \}_{k=1}^{\infty} \) in \( H \) such that \( \chi \circ \tau_{h_k} \to 1_N \) in the \( \ast \)-topology of \( L_\infty(N) \). Then, for every \( f \in L_\infty(G) \), \( f \neq 0 \), the \( \ast \)-closed subspace generated by the two-sided translates of \( f \) contains a character \( \psi \) of \( G \). Moreover, if \( f \) is uniformly continuous, the character \( \psi \) belongs to the \( \ast \)-closure of some norm-bounded set of linear combinations of two-sided translates of \( f \).
Proof. Let $M$ denote the $w^*$-closed subspace generated by the two-sided translates of the function $f \in L_{\infty}(G), f \neq 0$. The subspace $M$ contains all functions $g$ such that

$$g(n, h) = f(n_2 \tau_{h^n}(n_1) \tau_{h^n}(n), h'h''h)$$

where $n_1, n_2 \in N$ and $h', h'' \in H$.

Let $h'' = 1_H$ and $n_1 = 1_N$. Then, applying Wiener's Theorem for $N \times H$, we deduce that $\chi \otimes \Phi \in M$ for some $\chi \in \hat{N}$ and $\Phi \in \hat{H}$. (Here, $\chi \otimes \Phi$ denotes the function defined by $\chi \otimes \Phi(n, h) = \chi(n)\Phi(h).$) If we apply (1) to $\chi \otimes \Phi \in M$ where $n_1 = n_2 = 1_N$, $h' = 1_H$ and $h'' = h_k$, we have

$$ (\chi \circ \tau_{h_k}) \otimes \Phi \in M \quad (k = 1, 2, \ldots) $$

and

$$ (\chi \circ \tau_{h_k}) \otimes \Phi \longrightarrow 1_N \otimes \Phi $$

which is a character of $G$.

If $f$ is uniformly continuous, then by Beurling's Theorem for $N \times H$, the function $\chi \otimes \Phi$ belongs to the $w^*$-closure of some norm-bounded set of linear combinations of two-sided translates of $f$. Proceeding as above, we complete the proof of the theorem.

References


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