APPROXIMATION BY NONFUNDAMENTAL SEQUENCES OF TRANSLATES

R. A. ZALIK

Abstract. For functions \( f(t) \) satisfying certain growth conditions, we consider a sequence of the form \( \{ f(c_n - t) \} \), nonfundamental in \( L_2(R) \), and find a representation for those functions which are in the closure of its linear span. Some theorems concerning degree of approximation are also proved.

In [1], we found necessary and sufficient conditions for a sequence of the form \( \{ f(c_n - t) \} \) to be fundamental in \( L_2(R) \). In this paper, motivated by earlier research of L. Schwartz [2], and I. I. Hirschmann, Jr. [3] (see also J. Korevaar [4], W. A. J. Luxemburg and J. Korevaar [5, p. 35, Theorem 8.2], and Clarkson and Erdös [6]), we consider the nonfundamental case and find a representation of those functions which are in the \( L_2(R) \) closure of the linear span of \( \{ f(c_n - t) \} \). Our result applies to a different class of functions than those considered by the above mentioned authors. The techniques developed to attack this problem are also applied to find a lower bound for the \( L_2(R) \) distance from \( f(c - t) \) to the linear span of \( \{ f(c_r - t); r = 0, \ldots, n \} \), obtaining a result similar to [5, p. 31, Theorem 7.1], [4, p. 363, Theorem 4], or [6, p. 6, Theorem 2]. Finally, we also prove a Jackson type theorem valid for a class of continuous functions defined on a bounded interval.

In what follows, \( \{ d_n \} \) will be a sequence of distinct real numbers, satisfying the following conditions:

\[
|c_n^2 - c_r^2| > \rho |n - r| \quad (\rho > 0) \quad \text{and} \quad \sum' |c_n|^{-2} < \infty. \tag{1}
\]

(By \( \sum' |c_n|^{-2} \) we denote the sum of all terms of the form indicated, with nonvanishing denominator.) Note that (1) is satisfied if, for instance

\[
|c_{n+1}| > \rho |c_n| \quad (\rho > \sqrt{2} ).
\]

Given a function \( f(t) \), by \( F(t) \) we shall denote its Fourier transform. Thus

\[
F(t) = (2\pi)^{-1/2} \int_R \exp(\imath xt)f(t) \, dt.
\]

We shall assume that there are strictly positive numbers \( a, b \) and \( \alpha \), such that for \( t \) real, \( f(t) = O[\exp(-\alpha t^2)] \), \( F(t) = O[\exp(-\alpha t^2)] \), \( t \to \infty \), and \( \exp(-\beta t^2)/F(t) \) is in \( L_2(R) \). By a theorem of Babenko, later generalized by Gel'fand and Šilov, we know that the growth condition on \( f(t) \) can be replaced by the assumption that \( F \) is an

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entire function of order 2 and finite type (cf. [7, p. 238, Theorem 3]). Finally, if \( f_n(t) = f(c_n - t) \), and \( F_n(t) \) is its Fourier transform, it is readily seen that \( F_n(-t) = F(t) \exp(c_n it) \); we shall denote by \( S \) the linear span of the sequence \( \{ f_n \} \), and by \( T \) the linear span of the sequence \( \{ F_n \} \).

Our first result is:

**Theorem 1.** Assume \( \{ c_n \} \) satisfies (1). Then if the function \( g(t) \) is in the \( L_2(\mathbb{R}) \) closure of \( S \), it coincides a.e. on \( \mathbb{R} \) with a series of the form \( \sum b_j f_j(t) \).

Theorem 1 is proved with the help of the following auxiliary proposition:

**Lemma.** Assume \( \{ c_n \} \) satisfies (1). Then there are continuous functions \( p_k(t) = p_k(t, \mu) \), having Fourier transforms \( m_k(t) = m_k(t, \mu) \), satisfying the following conditions:

(a) Let \( h(t) = \exp(-\delta t^2) / |F(t)| \); then for every \( \mu < 1/(2b) \) and positive,
\[
|m_k(t, \mu)| < d \exp\left[ -\frac{1}{2} (1/(2\mu) - b) t^2 + \mu c_n^2 \right] h(t),
\]
where \( d \) is independent of \( k \).

(b) \( \int_R p_k(t) f_n(t) \, dt = \delta_{kn} \) where \( \delta_{kn} \) is Kronecker's delta.

(c) For \( g(t) \) in \( L_2(\mathbb{R}) \), let \( b_k(g) = \int_R p_k(t) g(t) \, dt \), then for any \( \delta < \alpha \) and positive, there is a value of \( \mu \) and a number \( \gamma \) such that for all real \( t \),
\[
|b_k(g)f_n(t)| < c^2 \|g\|_{L_2(\mathbb{R})} \exp(-\delta c_n^2 + \gamma t^2),
\]
where \( c \) is independent of \( k \), and if for this value of \( \mu \), \( S(g, t) = \sum b_n(g)f_n(t) \), then
\[
|S(g, t)| < M(t) \|g\|_{L_2(\mathbb{R})}, \quad \text{where } M(t) = c \exp(\gamma t^2) \sum \exp(-\delta c_n^2).
\]

Using the preceding Lemma, we can prove:

**Theorem 2.** Assume \( \{ c_n \} \) satisfies (1), and let \( c \) be any real number not in the range of the sequence \( \{ c_n \} \). If \( |c| = |c_n| \) for some \( n \), let \( m_c = 1 \); otherwise, let \( m_c = \inf |1 - (c/c_n)^2| \), the infimum being taken over the set of natural numbers. Let \( d_c \) denote the \( L_2(\mathbb{R}) \) distance from \( F(t) \exp(cti) \) to \( T \). Then there is a number \( D > 0 \), independent of \( c \) and \( k \), such that \( d_c > D m_c^2 \exp(-c^2/8b) \).

**Remark.** Since the Fourier transform is norm-preserving in \( L_2(\mathbb{R}) \), \( d_c \) also denotes the \( L_2(\mathbb{R}) \) distance from \( f(c - t) \) to \( S \). It should also be pointed out that the lower bound in Theorem 2 is not the best possible.

From Theorem 2 we obtain the following

**Corollary.** If \( c \) is not in the range of the sequence \( \{ c_n \} \), then neither \( F(t) \exp(cti) \) is in the \( L_2(\mathbb{R}) \) closure of \( T \), nor is \( f(c - t) \) in the \( L_2(\mathbb{R}) \) closure of \( S \).

Finally, we have:

**Theorem 3.** Assume that \( \{ c_n \} \) satisfies (1), and let \( g(t) \) be a function in the \( L_2(\mathbb{R}) \) closure of \( S \). Let \( (a_i, b_i) \) be a bounded interval, assume \( g(t) \) is continuous thereon, and let \( d_n \) denote the uniform distance from \( g(t) \) to the span of \( \{ f_r(t); r = 0, \ldots, n \} \) in \( (a_i, b_i) \). Then for any number \( \delta, 0 < \delta < \alpha \), there are numbers \( D \) (independent of \( n \) and \( g \)) and \( \gamma \) (independent of \( n \)), such that \( d_n < D \|g\|_{L_2(\mathbb{R})} \exp(-\delta \gamma) \).
We shall use the following notation: By $\Sigma^{(k)}$ and $\Pi^{(k)}$ we shall denote sums and products of the form indicated, $k$th term deleted. For the theory of entire functions we shall refer to the book by R. P. Boas, Jr. [8].

**Proof of Lemma.** We shall only consider the case in which $c_n \neq 0$ for all $n$, the other case being similar. Let $r_k(z) = \Pi^{(k)}(1 - z^2/c_n^2)$, and $\mu > 0$. As in the proof of [5, p. 33, Lemma 7.2] (with $h_n = c_n^2$), we see that the sequence $\{\exp[(\mu/4)c_n^2]|r_k(c_n)\}$ is bounded away from zero, say

$$\exp[(\mu/4)c_n^2]|r_k(c_n)| > D > 0. \quad (2)$$

Clearly $r_k(z) = P_k(z)P_k(-z)$, where

$$P_k(z) = \prod^{(k)}(z/c_n, 1) = \prod^{(k)}(1 - z/c_n) \exp(z/c_n).$$

If $n_k(r)$ denotes the number of elements in the sequence $\{c_n, n \neq k\}$ within the disk of radius $r$, and $n(r)$ is similarly defined for the whole sequence $\{c_n\}$, it is clear that $n_k(r) \leq n(r)$. In view of this inequality, setting $|z| = r$ and applying to $P_k(z)$ the same technique employed in the proof of [8, pp. 29–30, 2.10.13], we readily see there is a function $u(r)$ (the same for all $k$), such that $\lim_{r \to \infty} u(r) = 0$, and

$$|r_k(z)| < \exp[u(r)r^2] \quad (3)$$

for all complex $z$. Setting

$$q_k(z) = q_k(\mu, z) = (2\pi)^{-1/2} \exp[(\mu/4)(z^2 - c_n^2)]r_k(z)/r_k(c_n),$$

we see that

$$q_k(-c_n) = (2\pi)^{-1/2} \delta_{kn}. \quad (4)$$

In view of (2) and (3), a straightforward computation shows that

$$\int_R |q_k(x + yi)|^2 dx < d_1^2 \exp[(\mu)(y^2 + c_n^2)],$$

$$\int_R |(x + yi)q_k(x + yi)|^2 dx < d_2^2 \exp[(\mu)(y^2 + c_n^2)],$$

where $d_1$ and $d_2$ are independent of $k$ (they are, of course, dependent on $\mu$). Proceeding as in the proof of the necessity part of Theorem 3 in [1, pp. 304–305], we conclude that $q_k(z)$ is the Fourier transform of a function $h_k(t) = h_k(t, \mu)$ (i.e. $q_k(z) = (2\pi)^{-1/2} \int_R h_k(t) \exp(zt) dt$), such that $h_k(t)$ is continuous, and (for $t$ real),

$$|h_k(t)| < d \exp[- t^2/(2\mu) + \mu c_n^2], \quad (5)$$

where $d$ is independent of $k$. Let $\mu < 1/(2b)$. Then, if $m_k(t) = m_k(t, \mu) = h_k(t)/F(t)$, and bearing in mind that $h(t) = \exp(-bt^2)/F(t)$ is in $L_2(R)$ by hypothesis, it is clear from (5) that

$$|m_k(t)| < d \exp[- (1/(2\mu) - b)t^2 + \mu c_n^2]h(t). \quad (6)$$

Let $p_k(t)$ be the inverse Fourier transform of $m_k(t)$. By Plancherel’s formula and (6), we see that

$$\int_R |p_k(t)|^2 dt = \int_R |m_k(t)|^2 dt < c^2 \exp(2\mu c_n^2), \quad (7)$$

where $c$ is independent of $k$. 

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From Plancherel's formula and (5), we also see that
\[
\int_R p_k(t)f_n(t)\,dt = \int_R p_k(t)f(c_n - t)\,dt = \int_R m_k(t)F(t)\exp(-c_n t)\,dt \\
= \int_R h_k(t)\exp(-c_n t)\,dt = (2\pi)^{1/2}q_k(-c_n) = \delta_{kn}.
\]
We have thus shown that
\[
b_k(f_n) = \delta_{kn}. \tag{8}
\]
Let \(g(t)\) be a function in \(L_2(R)\). Applying the Cauchy-Schwartz inequality and (7), we see that
\[
|b_n(g)f(c_n - t)| \leq c\|g\|_{L_2(R)} \exp\left[\mu c_n^2 - \alpha(c_n - t)^2\right].
\]
Let \(\delta\) be any number such that \(0 < \delta < \alpha\). Setting \(\mu = \alpha - \delta - \epsilon\), where \(0 < \epsilon < \alpha - \delta\), we see that \(\mu c_n^2 - \alpha(c_n - t)^2 = -\delta c_n - \epsilon(c_n - t)^2 + \gamma t^2\), whence we conclude that for this value of \(\mu\),
\[
|b_n(g)f(c_n - t)| \leq c\|g\|_{L_2(R)} \exp(-\delta c_n^2 + \gamma t^2), \tag{9}
\]
whence we readily conclude that
\[
|S(g, t)| \leq M(t)\|g\|_{L_2(R)}, \tag{10}
\]
where \(M(t) = c\exp\gamma t^2 \sum \exp(-\delta c_n^2)\), and the conclusion follows from (6), (8), (9) and (10). Q.E.D.

**Proof of Theorem 1.** Assume that \(g(t)\) is in the \(L_2(R)\) closure of \(S\). Let \(\{g_n\}\) be a sequence of elements of \(S\) that converges to \(g(t)\) in the \(L_2(R)\) distance. Taking if necessary a subsequence thereof, we can assume without loss of generality that \(\{g_n\}\) converges to \(g(t)\) a.e. in \(R\).

From (8) we readily conclude that \(S(g_n, t) = g_n(t)\). Applying (10), we thus see that
\[
|g_n(t) - S(g, t)| = |S(g_n, t) - S(g, t)| \\
= |S(g_n - g, t)| \leq M(t)\|g_n - g\|_{L_2(R)}. \tag{11}
\]
Thus \(S(g, t) = \lim_{n \to \infty} g_n(t)\), and therefore \(g(t) = S(g, t)\), a.e., whence the conclusion follows. Q.E.D.

**Proof of Theorem 2.** Assume first that \(|c| \neq |c_n|\) for all \(n\). Since the sequence \(\{c_n\}\) diverges, there is a number \(k\) such that \(|c_k| < |c| < |c_{k+1}|\). Let \(d_n = c_n\) if \(n < k\), \(d_k = c\), and \(d_n = c_{n+1}\) if \(n > k\). Clearly (1) is also satisfied (with the same \(p\)) by the sequence \(\{d_n\}\). Let \(r(z) = \prod(1 - z^2/c_n^2)\), and \(P(z) = \prod^{(k)}(1 - z^2/d_n^2)\). Clearly, \(r(c) = (1 - c^2/c_k^2)(1 - c^2/c_{k+1}^2)P(c)\). Let \(\mu > 0\); inspection of the proof of [5, p. 33, Lemma 7.2] shows that
\[
\exp\left[\left(\mu/4\right)c^2\right]P(c) = \exp\left[\left(\mu/4\right)c^2\right]P(d_k) \geq D > 0
\]
(where \(D\) is independent of \(c\)), and therefore
\[
|r(c)| > m^2D. \tag{12}
\]
Let \(q(z) = q(\mu, z) = \exp[-(\mu/4)z^2]r(z)\), and \(0 < \mu < 1/(2b)\). Proceeding again as in [1, pp. 304–305], we see that
\[
q(z) = (2\pi)^{-1/2} \int_R m(t)F(t) \exp(zt) \, dt,
\]
where \(m(t) = m(\mu, t)\) is such that \(|m(t)| < d \exp[-(1/(2\mu) - b)t^2]h(t)\), and \(h(t) = \exp(-br^2)/|F(t)|\) is in \(L_2(R)\); thus the \(L_2(R)\) norm of \(m(t)\) is independent of \(c\). Since \(q(-c_n) = 0\), it readily follows from [9, p. 337, (V. 75)], that
\[
|q(c)| = \left| \int_R m(t)F(t) \exp(ct) \, dt \right| < d\|m\|_{L_2(R)}.
\]
Since \(\mu < 1/(2b)\), and (12) implies that \(|q(c)| > Dm_2^2 \exp[-(\mu/4)c^2]\), the conclusion follows. If \(|c| = c_k\) for some \(k\), define \(d_n = c_n\) if \(n \neq k\), and \(d_k = c = -c_k\). Thus if \(r(z)\) is defined as above, \(r(z) = (1 - z/c) \prod^{(k)}(1 + z/d_n)\), and therefore \(r(c) = 2 \prod^{(k)}(1 + d_k/d_n)\). Since the sequence \(\{d_n\}\) satisfies (1), the conclusion follows as above. Q.E.D.

**Proof of Theorem 3.** Let \(g(t)\) be a function in the \(L_2(R)\) closure of \(S\). From Theorem 1 we know that \(g(t) = S(g, t)\) a.e. on \(R\). However, it is readily seen from (9) and the continuity of the functions \(f_r(t)\), that \(S(g, t)\) is continuous on \(R\), and therefore identical with \(g(t)\) on \((a, b)\). Thus,
\[
g(t) = \sum_{r=0}^{\infty} b_r(g)f_r(t) \tag{13}
\]
thereon. From (9) and (1) we know that if \(t\) is in \((a, b)\), and \(\eta^2 = \sup\{a^2, b^2\}\), then
\[
|b_r(g)f_r(t)| < c\|g\|_{L_2(R)} \exp(\gamma r^2) \exp(-\delta c^2)
\leq c\|g\|_{L_2(R)} \exp(\gamma r^2 + c_0^2) \exp(-\delta r^2). \tag{14}
\]
Combining (13) and (14) we have
\[
d_n < |g(t) - \sum_{r=0}^{n} b_r(g)f_r(t)| < \sum_{r=n+1}^{\infty} |b_r(g)f_r(t)|
< Q\|g\|_{L_2(R)} \sum_{r=n+1}^{\infty} \exp(-\delta r^2)
= Q\|g\|_{L_2(R)} \left[ \exp(-\delta r^2) / (1 - \exp(-\delta r)) \right],
\]
whence the conclusion follows. Q.E.D.

**References**


Department of Mathematics, Auburn University, Auburn, Alabama 36830