ON THE DIMENSION OF INJECTIVE BANACH SPACES

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Abstract. The purpose of this note is to give an affirmative answer, assuming the generalized continuum hypothesis, to a problem of H. Rosenthal on the cardinality of the dimension on injective Banach spaces.

The problem in question is contained in [4, Problem 7.a]; in this connection we prove the following result.

Theorem 1. Assume the G.C.H. If X is an infinite dimensional injective Banach space with dim X = \( \alpha \), then \( \alpha^\omega = \alpha \).

We start with some preliminaries.

We denote cardinals by \( \alpha, \beta \); \( \omega \) denotes the cardinality of natural numbers. We denote by \( \alpha^\omega \) the cardinality of the family of countable subsets of \( \alpha \). For a cardinal \( \alpha \), we denote by \( \text{cf}(\alpha) \) the least cardinal \( \beta \) such that \( \alpha \) is the cardinal sum of \( \beta \) many cardinals, each smaller than \( \alpha \). A cardinal \( \alpha \) is regular if \( \alpha = \text{cf}(\alpha) \), and singular if \( \text{cf}(\alpha) < \alpha \). The least cardinal strictly greater than \( \beta \) is denoted by \( \beta^+ \). The cardinality of a set \( A \) is denoted by \( |A| \). The generalised continuum hypothesis (G.C.H.) is the statement that \( \alpha^+ = 2^\alpha \) for all infinite cardinals \( \alpha \).

A real Banach space \( X \) is injective if for every Banach space \( Y \) and every bounded linear isomorphism \( T: X \to Y \), there is a bounded linear projection \( P: Y \to T(X) \). If \( \Gamma \) is a set, we denote by \( I^1(\Gamma) \) the Banach space of real-valued functions on \( \Gamma \) which are absolutely summable. If \( X \) is a Banach space we denote with \( \text{dim} \ X \) the least cardinal \( \alpha \) such that there is a family \( F = \{ x_\xi : \xi < \alpha \} \) of elements of \( X \) with the property that \( X \) is the closed linear span of \( F \).

Lemma 2. Let \( X \) be an injective Banach space with \( \text{dim} \ X = \alpha \). Then \( I^1(\alpha) \) is isomorphic to a subspace of \( X^* \).

Proof. Since \( X \) is a complemented subspace of \( C(S) \) for some compact space \( S \), \( X^* \) is a complemented subspace of \( L^1(\lambda) \) for some measure \( \lambda \). So the conclusion is a direct consequence of Theorem 2.5 of [3].

Proof of Theorem 1. Let us assume that the conclusion is false. Then there is an injective Banach space \( X \) with \( \text{dim} \ X = \alpha \) and \( \alpha^\omega > \alpha \). Under the G.C.H., \( \alpha^\omega > \alpha \) means that \( \text{cf}(\alpha) = \omega \) and since \( I^\infty(\mathbb{N}) \) is isomorphic to a subspace of \( X \) [5] it follows that \( \alpha > \text{cf}(\alpha) \).

We choose a sequence \( \{ \alpha_\xi : \eta < \omega \} \) of regular cardinals such that \( \alpha_1 = \omega^+ \), \( \alpha_{\eta + 1} > 2^\omega \) and \( \sum_{\eta < \omega} \alpha_\eta = \alpha \).

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From Lemma 2 there is a family \( \{ e_\xi : \xi < \alpha \} \) of elements of the unit ball of \( X^* \) equivalent to the canonical basis for \( l^1(\alpha) \).

Let, also, \( \{ x_\xi : \xi < \alpha \} \) be a norm dense subset of \( X \). Using finite induction we choose a family \( \{ A_\eta : \eta < \omega \} \) of subsets of \( \alpha \) such that:

(i) \( A_\eta \subset \{ \xi : \alpha_\eta < \xi < \alpha_{\eta+1} \} \),
(ii) \( |A_\eta| > 2 \), and
(iii) for \( \eta < \omega \) and \( \xi_1, \xi_2 \in A_\eta \)

\[ e_\xi(x_\xi) = e_\xi(x_\xi) \text{ for all } \xi < \alpha_\eta. \]

For every \( \eta < \omega \) we choose \( \xi_1^\eta \neq \xi_2^\eta \) elements of \( A_\eta \), and we set \( e_\eta = e_{\xi_1^\eta} - e_{\xi_2^\eta} \). Then the sequence \( \{ e_\eta : \eta < \omega \} \) converges weak* to \( 0 \in X^* \), and since \( X \) is injective, \( \{ e_\eta : \eta < \omega \} \) is in fact weakly convergent [2]. On the other hand, \( \{ e_\eta : \eta < \omega \} \) is equivalent to the usual basis for \( l^1(\mathbb{N}) \), a contradiction.

**Remark 1.** As the referee has remarked, the proof shows immediately the following more general statement:

If \( X \) is an \( \mathcal{C}_\omega \) Grothendieck space, then under the G.C.H. we have \( (\dim X)^\omega = \dim X \). (Recall that a Banach space \( X \) is a Grothendieck space if every sequence in \( X^* \) which is weak* convergent necessarily converges weakly.)

**Remark 2.** We do not know what happens without any set-theoretical assumption. In this direction we proved in [1] the following.

**Theorem A.** If \( X \) is an injective Banach space in which each weakly compact subset is separable and \( \dim X = \alpha \) then \( \alpha^\omega = \alpha \).

**Theorem B.** Let \( \alpha \) be a cardinal and \( X \) be an injective Banach space such that \( l^1(\alpha) \) is isomorphic to a subspace of \( X \). Then \( X \) contains isomorphically a copy of \( l^1(\alpha^\omega) \).

**References**

1. S. Argyros, Weak compactness in \( L^1(\lambda) \) and injective Banach spaces (to appear).