RADII OF IMMERSED MANIFOLDS AND
NONEXISTENCE OF IMMERSIONS

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Abstract. Let \( M \) be a compact Riemannian manifold isometrically immersed in a complete Riemannian manifold \( N \). By the radius of \( M \) in \( N \), we mean the minimum of radii of closed geodesic balls in \( N \) which contain \( M \). Using the concept of a radius, we will give a theorem about the nonexistence of isometric immersions, which is a generalization of J. D. Moore's result.

1. Introduction. Let \( M \) and \( N \) denote \( C^\infty \) Riemannian manifolds, \( K \) and \( K^* \) their respective sectional curvature functions. J. D. Moore [3] proved that when \( N \) is a complete simply connected Riemannian manifold with \( a < K^* < b < 0 \), and \( M \) is a compact Riemannian manifold with \( K < a - b \), \( M \) possesses no isometric immersion in \( N \), unless \( \dim N > 2 \dim M \). On the other hand H. Jacobowitz [1] showed that an isometric immersion of an \( n \)-dimensional compact Riemannian manifold with sectional curvature always less than \( \lambda^{-2} \) into Euclidean space of dimension \( 2n - 1 \) can never be contained in a ball of radius \( \lambda \). In this note, using methods similar to those of [3], we generalize the results of J. D. Moore and H. Jacobowitz. At first we define a positive continuous function \( C(b, d) \) on \( (-\infty, 0) \times (0, \infty) \) by

\[
C(b, d) = \begin{cases} 
1 & \text{if } b = 0, \\
\frac{1}{d} & \text{if } b < 0, \\
\sqrt{-b} \coth (d\sqrt{-b}) & \text{if } b < 0.
\end{cases}
\]

This function is monotonically decreasing with respect to \( b \) and also \( d \). We will prove

Theorem 1. Let \( N \) be a complete simply connected Riemannian manifold whose sectional curvatures \( K^*(\sigma) \) satisfy the inequalities

\[
a < K^*(\sigma) < b < 0, \tag{1}
\]

\( M \) be a compact Riemannian manifold with diameter \( d \). Assume that at every point of \( M \), there is a \( p \)-dimensional subspace in the tangent space, along whose plane elements \( \sigma \), it holds

\[
K(\sigma) < a + C^2(b, d). \tag{2}
\]

If \( \dim N < \dim M + p \), then \( M \) cannot be isometrically immersed in \( N \).

As the function \( C(b, d) \) satisfies \( C^2(b, d) > -b \), this theorem strengthens the
result in [3]. Theorem 1 is immediate from the following theorem which generalizes H. Jacobowitz's result [1].

**Theorem 2.** Assume that N satisfies the same conditions as in Theorem 1. Let d be some positive constant. Let M be a compact manifold such that at every point of M, there is a p-dimensional subspace in the tangent space, along whose plane elements, the inequality (2) holds. If dim N < dim M + p, then no isometric immersion of M into N is contained in a ball of radius d.

2. **Radii of immersed manifolds.** We will deal with a ball containing an immersed manifold as in [1]. Let M be a compact Riemannian manifold isometrically immersed in a complete Riemannian manifold N. Let \( d(\cdot, \cdot) \) be the distance function of N. For any point \( x \in N \) and any \( r > 0 \), put \( B(x, r) = \{ y \in N, d(x, y) < r \} \). Then we set

\[
  r(M) = \inf \{ r; M \subset B(x, r) \}
  = \inf \{ \max \{ d(x, y), y \in M \}, x \in N \}. \tag{3}
\]

As M is compact, we can prove there is a point \( x_0 \in N \) such that \( B(x_0, r(M)) \supset M \). Moreover there is a point \( y_0 \in M \) such that \( d(x_0, y_0) = r(M) \). We will call \( r(M) \) the radius of M in N and \( B(x_0, r(M)) \) a minimum ball containing M. Generally, there are several minimal balls containing M. For example, let \( S^1 = \{ (x_1, x_2); x_1^2 + x_2^2 = 1 \} \) be naturally imbedded in \( S^2 = \{ (x_1, x_2, x_3); x_1^2 + x_2^2 + x_3^2 = 1 \} \). Then minimal balls containing \( S^1 \) are \( B((0, 0, 1), \pi/2) \) and \( B((0, 0, -1), \pi/2) \). But there is only one minimal ball for a compact manifold immersed in a euclidean space. In fact we have

**Theorem 3.** Let M be a compact manifold immersed in an n-dimensional euclidean space \( E^n \). Then there is only one point \( x_0 \in E^n \) such that \( r(M) = \max \{ d(x_0, y), y \in M \} \).

**Proof.** Take a point \( x_0 \) which satisfies the above equality. Let \( S(x_0, r(M)) = \{ y \in M, d(x_0, y) = r(M) \} \). At first we will prove that \( S(x_0, r(M)) \) contains more than one point. Suppose there is only one point \( y_0 \) in M with \( d(x_0, y_0) = r(M) \). Take a positive \( \delta \) satisfying \( \delta < r(M)/2 \). Put \( r_1 = \max \{ d(x_0, y), M - B(y_0, \delta) \supset y \} \). Then \( r_1 < r(M) \). Hence if \( x_1 \in B(x_0, \epsilon_1/2) \),

\[
  \max \{ d(x_1, y), M - B(y_0, \delta) \supset y \} < r_1 + \frac{\epsilon_1}{2} < r(M),
\]

where we put \( \epsilon_1 = r(M) - r_1 \). Let \( x_\epsilon = \epsilon x_0 + (1 - \epsilon)y_0 \). An easy calculation shows that for any \( y \in M \cap B(y_0, \delta) \),

\[
  d(x_\epsilon, y) < r(M)^2 - 2\epsilon\delta + \epsilon^2 < r(M)^2 \quad (0 < \epsilon < 2\delta).
\]

Thus if \( \epsilon < \min(\epsilon, 2\delta) \), \( \max \{ d(x_\epsilon, y), y \in M \} < r(M) \). This is contrary to the definition of \( r(M) \). Let \( \Pi(x_0) \) be the k-dimensional affine subspace of \( E^n \) spanned by \( S(x_0, r(M)) \), where \( 1 < k < n \). Take \( k + 1 \) points \( y_0, y_1, \ldots, y_k \) from \( S(x_0, r(M)) \) such that \( y_1 - y_0, y_2 - y_0, \ldots, y_k - y_0 \) are linearly independent. Then they obviously span \( \Pi(x_0) \). If \( k = n \), there is no point except \( x_0 \) whose distances from \( y_0, y_1, \ldots, y_n \) are \( r(M) \). Hence let \( 1 < k < n \). We will prove \( x_0 \in \Pi(x_0) \).
Suppose that \( x_0 \not\in \Pi(x_0) \). We may assume that \( x_0 = 0 \) and \( \Pi(x_0) = \{(a, x_2, \ldots, x_{k+1}, 0, \ldots, 0)\} \), where \( a \) is a positive constant. Set \( I = \{(x_1, \ldots, x_n); \ 3 \ a < x_1\} \). Then \( S(x_0, r(M)) \subset I \) and \( M - I \) is compact. Hence we have \( \max\{d(x_0, y), y \in M - I\} = r_1 < r(M) \). Let \( x_0 = (\epsilon, 0, \ldots, 0) \in \mathbb{R}^n \) \((0 < \epsilon < a)\). Then we can prove

\[
\max\{d(x_0, y), y \in M \cap I\} < \sqrt{r(M)^2 - \frac{\epsilon^2}{2}} < r(M).
\]

Let \( 0 < \epsilon < \min((r(M) - r_1)/2, a) \). It follows that

\[
\max\{d(x_0, y), y \in M - I\} < r(M) - \frac{\epsilon}{2} < r(M).
\]

Thus we obtain \( \max\{d(x_0, y), y \in M\} < r(M) \). But this is contrary to our assumption. Now, \( x_0 \in \Pi(x_0) \) whose distances from \( y_0, \ldots, y_k \) are \( r(M) \) as similarly as in the case \( k = n \). Lastly we assume that there is a point \( x_1 \in \Pi(x_0) \) with \( B(x_1, r(M)) \supset M \). But this is impossible because it follows from this assumption that at least one of \( d(x_1, y_i) \) \((i = 0, 1, \ldots, k)\) should be greater than \( r(M) \). The proof is now completed exactly.

3. Proofs of Theorems 1 and 2. Firstly we assume that \( M \) and \( N \) satisfy the hypotheses of Theorem 2. Moreover suppose that \( M \) is isometrically immersed in \( N \) and contained in a ball of radius \( d \). Then we have \( r(M) < d \). Take \( x_0 \in N \) and \( y_0 \in M \) satisfying \( r(M) = d(x_0, y_0) \). Let \( \gamma : [0, 1] \rightarrow N \) be a minimal geodesic with \( \gamma(0) = x_0, \gamma(1) = y_0 \). For each unit tangent vector \( v \in T_{y_0}M \), there is a unique Jacobi field \( V \) along \( \gamma \) such that \( V(0) = 0, V(1) = v \). Corresponding to \( V \), we have a one-parameter family of geodesics from \( x_0 \) to \( M \), \( \gamma_s(t) = \gamma(s, t): (-\epsilon, \epsilon) \times [0, 1] \), which satisfies \( \gamma_0(t) = \gamma(t), \ (\partial \gamma(s, t)/\partial s)(0, t) = V(t) \). We set

\[
L(\gamma_s) = \frac{1}{2} \int_0^1 \langle \gamma'_{s}, \gamma'_{s} \rangle \, dt.
\]

Then from the definition of \( \gamma \), it follows that \( L(\gamma_s) < L(\gamma) \). Hence

\[
0 > \left( \frac{d^2}{ds^2} L(\gamma_s) \right)_{s=0} = I(V, V) + \langle \alpha(v, v), \gamma'(1) \rangle,
\]

where \( \alpha \) is the second fundamental form of \( M \) in \( N \), and \( I(, \ ) \) is the index form. Taking a proper Jacobi field on a space of constant curvature \( b \), J. D. Moore [2] proved that \( I(V, V) > r(M)C(b, r(M)) \). Hence from (4) we get \( \langle \alpha(v, v), \gamma'(1) \rangle < -r(M)C(b, r(M)) \). Since \( \|\gamma'(1)\| = r(M) \), we obtain for all unit vectors \( v \in T_{y_0}M \), \( \|\alpha(v, v)\| > C(b, r(M)) \). On the other hand, if \( \sigma \) is a plane element which is spanned by \( v, w \) and satisfies (2), it holds that

\[
\langle \alpha(v, v), \alpha(w, w) \rangle - \|\alpha(v, v)\|^2 = K(\sigma) - K^*(\sigma) < C^2(b, d).
\]

Since the function \( C(b, d) \) is monotonically decreasing, it follows that \( C(b, d) < C(b, r(M)) \). Thus the proof of Theorem 1 is finished by the following lemma which was proved essentially by T. Otsuki [4]. We will prove by an argument due to T. A. Springer [2, Chapter 8, §4].
**Lemma.** Let $\alpha: \mathbb{R}^p \times \mathbb{R}^p \to \mathbb{R}^m$ be a symmetric bilinear mapping and $\langle \cdot, \cdot \rangle$ be a positive definite inner product on $\mathbb{R}^m$. If there is a nonnegative constant $C$ which satisfies
\[
\langle \alpha(v, v), \alpha(w, w) \rangle - \langle \alpha(v, w), \alpha(v, w) \rangle < C^2, \quad \langle \alpha(v, v), \alpha(v, v) \rangle > C^2,
\]
or
\[
\langle \alpha(v, v), \alpha(w, w) \rangle - \langle \alpha(v, w), \alpha(v, w) \rangle < C^2, \quad \langle \alpha(v, v), \alpha(v, v) \rangle > C^2, \quad \alpha(v, v) \neq 0,
\]
for all nonzero $v, w \in \mathbb{R}^p$, then we have $m \geq p$.

**Proof.** We extend $\alpha$ to a symmetric complex bilinear mapping of $\mathbb{C}^p \times \mathbb{C}^p \to \mathbb{C}^m$. The equation $\alpha(z, z) = 0$ is equivalent to a system of $m$ quadratic equations. If $m < p$, then this system of $m$ equations has a nonzero solution $z = x + \sqrt{-1} y$, where $x, y \in \mathbb{R}^p, y \neq 0$. As $\alpha(y, y) \neq 0$, from $\alpha(z, z) = 0$, we get $\alpha(x, x) = \alpha(y, y) \neq 0$ and $\alpha(x, y) = 0$. This is contrary to our assumption.

Theorem 1 follows easily from Theorem 2. Let $M$ be a compact Riemannian manifold with diameter $d$. If $M$ is isometrically immersed in $N$, it should be contained in some ball of radius $d$. Thus we have Theorem 1.

**References**


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