ON RECURRENCE OF A RANDOM WALK IN THE PLANE

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Abstract. The purpose of this note is to establish a sufficient condition for recurrence of a random walk \((S_n)\) in \(\mathbb{R}^2\). It follows from it that if \(S_n/n^{1/2}\) is asymptotically normal then we have recurrence.

Let \(X_1, X_2, \ldots\) be independent, identically distributed random variables in \(\mathbb{R}^k, k \geq 1\), with common distribution \(F\), and let for \(n > 1\), \(S_n = \sum_1^n X_i, S_0 = 0\). The random walk \(S = (S_n)_{n=0}^\infty\) has a point of recurrence at \(x\) if, for every \(\epsilon > 0\),

\[
P(\|S_n - x\| < \epsilon \text{ i.o.}) = 1.
\]

(1)

It is well known that the set of recurrence points is either empty or equals the smallest closed additive group containing the support of \(F\), see [1] or [3, §8.3]. In the latter case we say that \(S\) is recurrent. Also well known is the following criterion: \(S\) is recurrent if and only if

\[
\sum_0^\infty P(|S_n| < \epsilon) = \infty
\]

(2)

for some \(\epsilon > 0\), see the references above, and (2) holds if and only if

\[
\lim_{r \to 1} \int_A \Re \frac{1}{1 - rf(t)} \, dt = \infty
\]

(3)

for all neighborhoods \(A\) of 0, where \(f\) is the characteristic function of \(F\): the criterion (3) is Theorem 3 in [1].

The study of recurrence has been carried out, to a large extent, by using (3) and related criteria. For example, in one dimension \(E[X_i] = 0\) implies recurrence, which is rather easily deduced from (3). In [2] a probabilistic (combinatorial) proof is given that the weaker assumption \(S_n/n \to 0\) is sufficient for recurrence. In that paper it is also claimed that if, in \(\mathbb{R}^2\), \(S_n/n^{1/2}\) is asymptotically normal, which is the case when \(E[X_i] = 0\) (zero vector) and \(E[|X_i|^2] < \infty\), then we have recurrence. However, the argument indicated for this result in [2], and also in [3, Problem 14, p. 274], is misleading to say the least. This was discovered by students in Chung's class in 1977 and was first corrected by him then: it is the main purpose of this note to settle this matter.

For \(y = (y_1, \ldots, y_k) \in \mathbb{R}^k, k \geq 2\), we let \(|y| = \max_1 < i < k |y_i|\) throughout.
Proposition. Consider a random walk \( S = (S_n) \) in \( \mathbb{R}^2 \). If there exists an increasing, continuous function \( h > 0 \) on some interval \([0, c]\) such that 
\[
\int_0^c (h(u)/u) \, du = \infty \text{ and } \lim_{n \to \infty} P(|S_n| < x \cdot n^{1/2}) > x^2 \cdot h(x) \quad (4)
\]
for each \( x \in [0, c] \), then \( S \) is recurrent.

Proof. Since \( x^2 \cdot h(x) \) is uniformly continuous on \([0, c]\), and since the functions \( \inf_{n > k} P(|S_n| < x \cdot n^{1/2}) \) increase with \( x \), the inequality in (4) holds uniformly in \( x \) in the sense that for each \( \epsilon > 0 \) there exist \( n(\epsilon) \) such that
\[
\inf_{n > n(\epsilon)} P(|S_n| < x \cdot n^{1/2}) > x^2 \cdot h(x) - \epsilon \quad (5)
\]
for all \( x \in [0, c] \). Furthermore, for integers \( m > 1 \) we have
\[
\sum_{0}^{\infty} P(|S_n| < m) < 4m^2 \cdot \sum_{0}^{\infty} P(|S_n| < 1).
\]
This inequality was proved in [3, Lemma 1, p. 268] for \( \mathbb{R}^1 \) with the constant \( 2m \) on the right side. The same argument yields the result for \( \mathbb{R}^2 \) with the constant \( (2m)^2 \) due to our definition of \( S_n \) indicated above. By virtue of (2), it is hence sufficient to prove
\[
\lim_{m \to \infty} m^{-2} \sum_{0}^{\infty} P(|S_n| < m) = \infty.
\]
Fix \( C > 0 \) arbitrarily large. Take \( B > 0 \) so large that
\[
h(u) du > C.
\]
Let \( \epsilon > 0 \) be so small that
\[
\inf_{n > n(\epsilon)} P(|S_n| < x \cdot n^{1/2}) > x^2 \cdot h(x)/2
\]
for \( B^{-1} < x < c \), which is possible because of (5) and the monotonicity of \( x^2 \cdot h(x) \). Now, if \( n > n(\epsilon) \) and \( B^{-1} < m \cdot n^{-1/2} < c \), we have \( P(|S_n| < m) > m^2 \cdot n^{-1} \cdot h(m \cdot n^{-1/2})/2 \), so
\[
\lim_{m \to \infty} m^{-2} \sum_{0}^{\infty} P(|S_n| < m) > \lim_{m \to \infty} m^{-2} \cdot \sum_{n > n(\epsilon)} P(|S_n| < m)_{B^{-1} < m \cdot n^{-1/2} < c}
= \frac{1}{2} \lim_{m \to \infty} \int_{m^2 \cdot c^{-2}}^{B^2 m^2} \frac{h(m \cdot x^{-1/2})}{x} \, dx = \frac{1}{2} \int_{c^{-2}}^{B^2} \frac{h(x^{-1/2})}{x} \, dx = \int_{B^{-1}}^{c} \frac{h(u)}{u} \, du \geq C.
\]
Since \( C \) is arbitrary, \( \sum_{0}^{\infty} P(|S_n| < 1) = \infty \) and hence \( S \) is recurrent.  

If \( S_n/n^{1/2} \) is asymptotically normal, the Proposition renders \( S \) recurrent: simply let \( c = 1 \) and let \( h(x) \) be constant = \( \min_{|y| < 1} g(y) \), where \( g \) is the relevant normal density.
It may be noticed that for every $\beta < 2$ there is a $F$ such that $E[|X_1|^\beta] < \infty$ but $S$ is transient. Namely, let $X_i = (X'_i, X''_i)$ where the variables $X'_i, X''_i$ are independent, stable and symmetric with index $\alpha, \beta < \alpha < 2$: such a distribution has continuous density and all moments of order $< \alpha$ finite, see [4, Lemma 2, p. 545]. With $S'_n = \sum X'_i$, we obtain

$$P(|S'_n| < 1) = P(|S''_n| < 1)^2 = P(|S'_n \cdot n^{-1/\alpha}| < n^{-1/\alpha})^2$$

$$= P(|X'_i| < n^{-1/\alpha})^2 < 2 \cdot \gamma \cdot n^{-2/\alpha},$$

where $\gamma$ is the supremum of the density of $F$ on the interval $[-1, 1]$. Hence, $\Sigma_0^\infty P(|S_n| < 1)$ is finite, $S$ is transient.

With a function $h$ as in the Proposition, if $\lim_{n \to \infty} P(|S_n| < x \cdot n) > x \cdot h(x)$ for each $x \in [0, c]$ for a one-dimensional random walk $S$, then $S$ is recurrent. This sufficient condition covers the result by Chung and Ornstein, but also, for example, the case when $F$ is a Cauchy distribution such that $S''_n/n$ is distributed like $X'_1$: then $h(x)$ = a suitable constant will do.

Since $h$ is allowed to tend to 0 as $x \to 0$, the question arises whether we can find a distribution with slightly heavier tails than that of the Cauchy distribution so that $P(|S_n| < x \cdot n) \to 0$ for all $x > 0$ as $n \to \infty$, and still have recurrence: it turns out that a symmetric distribution on the integers with $P(X = k) = \gamma \cdot \log(1 + |k|) \cdot (1 + k^2)^{-1}, \gamma$ a normalizing constant, is such a distribution. To prove this, Fourier methods seem inevitable.

In order to illuminate that the condition $\int_0^c (h(u)/u)du = \infty$ is crucial, let $h > 0$ be increasing and such that $\int_0^c (h(u)/u)du < \infty$ for some $c > 0$ and suppose that

$$\lim_{n \to \infty} \sup_{0 < x < c} P(|S_n| < x \cdot n^{1/2})/x^2 \cdot h(x) < \infty$$

for a random walk $S$ in $R^2$. Then $S$ is transient, as is readily verified.

**REFERENCES**


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