HOMOTOPY AND UNIFORM HOMOTOPY. II

ALLAN CALDER AND JERROLD SIEGEL

Abstract. An elementary proof of the Bounded Lifting Lemma is given, together with a proof that homotopy and uniform homotopy do not agree for maps into compact spaces with infinite fundamental groups even though they can agree for maps into a noncompact space with infinite fundamental group.

The purpose of this paper is two-fold: (1) To give a short, unified and a great deal more transparent proof of the main geometrical results of [1], upon which all of [1] and [2] depend. (2) To give a proof that if $Y$ is compact and $\pi_1 Y$ is infinite then $[\beta, Y]$ is not a homotopy functor. It follows that the result of [1] concerning the relation between homotopy and uniform homotopy for finite-dimensional normal spaces is best possible. We wish to thank J. Keesling for his observation with regard to (2).

A fibration $p: E \to B$, by which we will mean a (Hurewicz) fibration such that $B$ has a numerable covering $\{U_n\}$ with $p^{-1}(U_n)$ trivial in the sense of Dold [4], is said to have the bounded lifting property (BLP) with respect to a subcategory $\mathcal{T}$ of $\mathcal{T} \otimes \mathcal{P}$, the category of topological spaces and maps, if for every space $X$ in $\mathcal{T}$ and map $f: X \to E$ such that $pf$ is bounded there exist a bounded map $g: X \to E$ which is homotopic to $f$ over $p$. (A bounded map is one for which the closure of the image is compact.) That is to say that any lift to $E$ of a bounded map into $B$ is homotopic over $p$ to a bounded map. We say $p$ has BLP($\mathcal{T}$).

Theorem 1 [1, (2.3) and (3.3)]. Let $F$ be the fiber of $p: E \to B$; then (1) if $F$ has the homotopy type of a compact space then $p$ has BLP($\mathcal{T} \otimes \mathcal{P}$), (2) if $F$ has the homotopy type of a CW-complex of finite type (i.e. finitely many cells in each dimension) then $p$ has BLP(fdNorm). Here fdNorm denotes the category of finite dimensional normal spaces.

A space $Y$ is said to have the relative compressibility property (RCP) with respect to $\mathcal{T}$ if for any space $X$ in $\mathcal{T}$, subspace $A$ of $X$ and map $f: X \to Y$ such that $f(A)$ is compact, there exists a homotopy $H: X \times I \to Y$ such that $H_0 = f$ and $H((X \times \{1\}) \cup (A \times I))$ is compact. We say that $Y$ has RCP($\mathcal{T}$).

Clearly, a compact space has RCP($\mathcal{T} \otimes \mathcal{P}$) and if $Z$ has RCP($\mathcal{T}$) and $Z$ dominates $Y$ (or in particular if $Z$ is homotopically equivalent to $Y$) then $Y$ has RCP($\mathcal{T}$). So the theorem will be a consequence of the following two lemmas.
Lemma 1. If $T$ is closed under closed subspaces and $F$ has $RCP(\mathbb{S})$ then $p$ has $BLP(\mathbb{S})$.

Lemma 2. A CW-complex of finite type has $RCP(fd\text{Norm})$.

Proof of Lemma 1. Let $X$ be in $\mathbb{S}$ and $f: X \to E$ a map such that $h = pf$ is bounded. By restricting to $h(X)$ if necessary we may assume that $B$ is compact. By our definition of fibration, there exists a finite open cover $\{U_i\}_{i=1}^n$ of $B$ such that $p^{-1}(U_i)$ is fiber homotopy equivalent to $U_i \times F$. Let $\phi_i$ be such a homotopy equivalence and $\psi_i$ its inverse.

Let $\{V_i\}$ be an open covering of $B$ such that $\overline{V_i} \subset U_i$. Put $E_i = h^{-1}(\overline{U_i})$ and $F_i = h^{-1}(\overline{V_i})$. Further, let $G_i: p^{-1}(\overline{U_i}) \times I \to p^{-1}(\overline{U_i})$ be a fiber homotopy from the identity to $\psi_i\phi_i$ and $\eta_i: B \to I$ be a map such that $\eta_i(B - U_i) = \{0\}$ and $\eta_i(\overline{V_i}) = \{1\}$.

Suppose that we have defined $g_{i-1}: X \to E$ such that $g_{i-1}$ is homotopic to $f$ over $p$ and $g_{i-1}(\bigcup_{j<i} F_j)$ is compact. Let $A = E_i \cap \bigcup_{j<i} F_j$ and let $H_i: E_i \times I \to U_i \times F$ be a fiber homotopy such that $H_i(x, 0) = \phi_i g_{i-1}$ and $H_i((E_i \times \{1\}) \cup (A \times I))$ is compact. Such $H_i$ exist since $F$ has $RCP(\mathbb{S})$ and $\overline{U_i}$ is compact.

Define $g_i: A \to F$ by

$$g_i(x) = \begin{cases} G_i(g_{i-1}(x), 2\eta_i h(x)), & \eta_i h(x) \in [0, \frac{1}{2}], \\ \psi_i H_i(x, 2\eta_i h(x) - 1), & \eta_i h(x) \in [\frac{1}{2}, 1]. \end{cases}$$

Then $g_i$ is homotopic to $g_{i-1}$ (and hence to $f$) over $p$ and $g_{i-1}(\bigcup_{j<i} F_j)$ is compact as it is contained in $g_{i-1}(\bigcup_{j<i} F_j) \cup \psi_i H_i((X \times \{1\}) \cup (A \times I))$. Putting $g_0 = f$, the result follows by induction up to $n$.

Proof of Lemma 2. Let $Y$ be a CW-complex of finite type and let $\phi: Y \simeq K: \psi$ be a homotopy equivalence and its inverse, where $K$ is a locally finite simplicial complex.

Suppose that $X$ is a finite-dimensional normal space, $A$ a subspace of $X$ and $f: X \to Y$ a map such that $\overline{f(A)}$ is compact. Let $\mathbb{V}$ be the star cover of $K$ and $\mathbb{U}$ a finite-dimensional cover of $X$ that refines $(\phi \circ f)^{-1}\mathbb{V}$. Let $\pi: X \to \nu\mathbb{U}$ be a canonical projection of $X$ onto the nerve of $\mathbb{U}$. Then there exists a simplicial map $\sigma: \nu\mathbb{U} \to K$ such that $\sigma\pi$ is contiguous to $\phi f$.

Let $\Theta: X \times I \to K$ be the linear deformation (see [3, p. 354]), $\phi f$ to $\sigma\pi$ then $\Theta(A \times I) \cup \sigma\pi(X)$ is contained in some $m$-skeleton $K^m$ of $K$. Let $D: Y \times I \to Y$ be a homotopy from the identity to $\psi\phi$. Define $H: X \times I \to Y$ by

$$H(x, t) = \begin{cases} D(f(x), 2t), & t \in [0, \frac{1}{2}], \\ \psi\Theta(x, 2t - 1), & t \in [\frac{1}{2}, 1]. \end{cases}$$

We may assume that $\psi$ is cellular so that $\psi(K^m) \subset Y^m$, the $m$-skeleton of $Y$, which is compact. Hence $H((X \times \{1\}) \cup (A \times I))$ is contained in a compact subset of $Y$, namely $Y^m \cup D(\overline{f(A)} \times I)$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Remarks. 1. A slight modification of the proof of Lemma 2 shows that for any given $X$ one only needs that $F$ has a compact $(\dim X)$-skeleton.

2. The nature of the proof of Lemma 2 seems to indicate that Theorem 1 contains all the useful geometric information about the relation between homotopy and uniform homotopy in that it shows that it is very unlikely that there are other useful categories $\mathcal{S}$ and $\mathcal{T}$ such that all the spaces in $\mathcal{S}$ have $\text{RCP}(\mathcal{T})$.

3. Part 2 of Theorem 1 is slightly stronger than (3.3) of [1] in that we do not require that $B$ has the homotopy type of a CW-complex.

As usual $\beta$ will denote the Stone-Cech compactification functor on the category of completely regular Hausdorff spaces.

**Theorem 2.** If $Y$ is compact and $\pi_1 Y$ is infinite then there is a homotopically nontrivial map from $\beta R$ to $Y$. Hence $[\beta -, Y]$ is not a homotopy functor on any category that contains the real line $\mathbb{R}$.

**Proof.** Let $PY$ denote the space of paths in $Y$ starting at $\ast \in Y$ and $p: PY \to Y$ the map $p(\lambda) = \lambda(1)$. Then $p$ is a fibration with fiber $\Omega Y$, the space of loops at $\ast$. That a map $\beta f: \beta R \to Y$ is homotopically trivial is equivalent to being able to factor it through $p$. This in turn is equivalent to being able to factor $f: \mathbb{R} \to Y$ through $p$ via a bounded map into $PY$, [1].

Since $\pi_1 Y$ is infinite, $\Omega Y$ has infinitely many path components. Let $\{\sigma_i\}_{i=0}^{\infty} \subset \Omega Y$ be such that $\sigma_0$ is the constant loop to $\ast$ and $\sigma_i$ and $\sigma_j$ are in distinct path components for $i \neq j$. Define $f: \mathbb{R} \to Y$ by $f(x) = \sigma_i \sigma_j^{-1}(x - i)$, $x \in [i, i + 1]$ and $f(x) = \ast$, $x < 1$. Since $Y$ is compact $f$ extends to $\beta R$.

Now any lift $\phi$ of $f$ to $PY$ must be unbounded as $\phi(i)$ and $\phi(j)$ must be in distinct path components of $\Omega Y$.

**Remark 4.** The condition that $Y$ is compact is essential in Theorem 2, since by [2, Theorem 3.4] for torsion abelian groups $G$, $[\beta -, K(G, 1)]$ is a homotopy functor on completely regular Hausdorff spaces, where $K(G, 1)$ is an Eilenberg-Mac Lane space of type $(G, 1)$. In particular one could take $G = \mathbb{Q}/\mathbb{Z}$.

**References**


**Department of Mathematics, Birbeck College, London WC1, England**

**Department of Mathematics, University of Missouri, St. Louis, Missouri 63121** (Current address of Jerrold Siegel)

**Current address** (Allan Calder): Department of Mathematics, New Mexico State University, Las Cruces, New Mexico 88003