BOUNDARY PRESERVING MAPS OF 3-MANIFOLDS

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Abstract. We prove an extension of Waldhausen’s theorem [5] conjectured by Hempel in [3].

We prove the following extension of Waldhausen’s theorem [5]:

Theorem 1. Let $M, N$ be $P^2$-irreducible 3-manifolds. Suppose that $M$ is compact, sufficiently large and $f : (M, \partial M) \to (N, \partial N)$ is a continuous map inducing an injection $f_* : \pi_1(M) \to \pi_1(N)$. Then, there is a proper homotopy $f_t : (M, \partial M) \to (N, \partial N)$ such that $f_0 = f$ and either

(i) $f_t : M \to N$ is a covering map, or

(ii) $M$ is an $I$-bundle over a closed surface, and $f_1(M) \subset \partial N$, or

(iii) $N$ (hence also $M$) is a solid torus or a solid Klein bottle and $f_1 : M \to N$ is a branched covering with branch set a circle, or

(iv) $M$ is a cube with handles and $f_1(M) \subset \partial N$.

If $f|B : B \to C$ is already a covering map, we may assume $f_t|B = f|B$, for all $t$ (where $B$ is any component of $\partial M$ and $C$ the component of $\partial N$ containing $f(B)$).

This theorem is conjectured in [3], where it is proved under additional restrictions. When $M, N$ are orientable and $f_*$ is an isomorphism, a variant of this result is proved by Evans in [2]. Our argument yields a simple proof of his result too. We refer to [1] and [4] for the concepts ‘geometric degree’, ‘absolute degree’, ‘orientation-true’, etc. The term ‘degree of a map $f$’ will be used for the twisted degree of $f$ and will be denoted by $\text{deg } f$ as in [4].

Lemma 2. Let $M$ be a compact irreducible 3-manifold and let $f : (M, \partial M) \to (N, \partial N)$ be a map into any aspherical 3-manifold such that $f_* : \pi_1(M) \to \pi_1(N)$ is injective. Let $S$ be a component of $\partial M$, $S'$ the component of $\partial N$ containing $f(S)$. If the geometric degree of $(f|S) : S \to S'$ is zero, then $M$ is a cube with handles and $f$ is properly homotopic to a map into $\partial N$.

Remark. If $(f|S)_* : \pi_1(S)$ is free subgroup of $\pi_1(S')$, then the geometric degree of $f|S$ is zero.

Proof. Since the geometric degree of $(f|S)$ is zero, $(f|S)$ is homotopic to a map of $S$ into $S' - p$ for any $p \in S'$. Hence $(f|S)_* : \pi_1(S) = H$ is a finitely generated free subgroup of $\pi_1(S' - p)$. Representing $H$ by the fundamental group of a wedge $X$ of circles, we can write $(f|S)$, up to homotopy, as a composite $S \to X \to S' - p$.

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Making \( g \) transverse to a point of \( X \) (\( \neq \) base point of \( X \)) we can find a nontrivial simple loop \( l \) in \( S \) such that \( g(l) \) is null-homotopic in \( X \) and hence \( f(l) \) is null-homotopic in \( S' - p \). Since \( f_*: \pi_1(M) \to \pi_1(N) \) injective, \( l \) bounds a disc \( D \) in \( M \). Split \( M \) along \( D \) to obtain \( M_1, M_2 \) (one of which may be empty) and maps \( g_i: M_i \to N \). Since \( f(l) \) is null-homotopic in \( S' - p \) and since \( N \) is aspherical we may assume that \( g_i \) maps \( (M_i, \partial M_i) \) to \( (N, \partial N) \), \( i = 1, 2 \). If \( S_i \) is the component of \( \partial M_i \) obtained from \( S \), \( g_i \) maps \( S_i \) into \( S' - p \) and therefore the degree of \( (g_i|S_i) \) is zero. Induction completes the proof of Lemma 2.

The following two lemmas follow easily from [1] and [4]; we sketch the proofs:

**Lemma 3.** Let

\[
\begin{array}{ccc}
(M, \partial M) & \xrightarrow{j} & (N, \partial N) \\
p \downarrow & & \downarrow q \\
(M, \partial M) & \xrightarrow{f} & (N, \partial N)
\end{array}
\]

be a commutative diagram, where \( p, q \) are covering projections with the same finite number of sheets and \( f_*: \pi_1(M) \to \pi_1(N) \) is surjective. Then the geometric degree of \( f \) is equal to the geometric degree of \( \tilde{f} \).

**Proof.** Let \( G(f), A(f) \) and \( a(f, 2) \) denote the geometric degree, absolute degree, and the mod 2 degree of \( f \) and similarly for \( \tilde{f} \). By a theorem of Hopf (see [1]), \( G(f) = A(f) \). By Theorem 3.1 of [1], \( A(f) = a(f, 2) \) mod 2. From the definition of geometric degree it is immediate that \( G(f) \geq G(\tilde{f}) \) and \( G(f) = G(\tilde{f}) \) mod 2. This together with the above assertions implies that \( G(f) = G(\tilde{f}) \) if \( G(f) < 1 \). Hence, it remains to consider the case when \( G(f) > 1 \). From the classification into three types used in defining absolute degree (see [1, p. 371] and [4, p. 375]), if \( f \) is of type II or III, then \( G(f) = A(f) \) \(< 1 \). Since \( G(f) > 1 \), \( f \) has to be of type I and hence it is orientation-true. In this case, by Theorem VIII of [4], \( A(f) = |\deg f| \). By Lemma 3.6 of [4],

\[
|\deg f| |\deg p| = |\deg ( f\tilde{p} )| = |\deg ( q\tilde{f}^\ast)| = |\deg q| |\deg \tilde{f}|.
\]

Since, by assumption \( |\deg p| = |\deg q| \), we have \( |\deg f| = |\deg \tilde{f}| \) and therefore \( G(f) = G(\tilde{f}) \) in the last case too.

**Lemma 4.** Let \( f: (M, \partial M) \to (N, \partial N) \) be a map inducing isomorphisms in the fundamental groups. If the geometric degree of \( f \) is nonzero, then \( f_*: \pi_1(M) \to \pi_1(N) \) is orientation-true (and the modulus of the degree of \( f \) is equal to the geometric degree of \( f \)).

**Proof.** Since \( G(f) \neq 0 \) and since \( f \) has no kernel, \( f \) is of type I and hence is orientation-true. Hence \( G(f) = |\deg f| \) by Theorem VIII of [4].

Lemma 2 is what is needed to extend the argument of [3] to the general case; we will quickly sketch the proof along the lines of proof of Theorem 13.6 of [3].
Proof of Theorem 1. Since we are not assuming that $N$ is compact, the theorem will follow from the case when $f_*$ is an isomorphism. From now on we will assume that $f_*$ is an isomorphism. We will show that either (ii) or (iii) or (iv) of Theorem is valid or $f$ is orientation-true and the degree of $f$ is $\pm 1$. Then by the extension of Lemma 3.1 of [2] to the nonorientable case, Theorem 1 follows.

By Lemma 2, either (iv) is valid or $(f|S)_* : \pi_1(S) \to \pi_1(S')$ has nonzero geometric degree for each component $S$ of $\partial M$; here $S'$ denotes the component of $\partial N$ containing $f(S)$. In the latter case $(f|S)_* \pi_1(S)$ is a subgroup of finite index in $\pi_1(S')$. Thus, if two components $S$, $T$ of $\partial M$ map into $S'$, then $(f|S)_* \pi_1(S)$ and $(f|T)_* \pi_1(T)$ intersect in a subgroup of finite index. Now one sees as in [3], that $M$ is a product $I$-bundle and (ii) is valid. Thus, either (iv) or (ii) is valid or $f$ satisfies

(p.1) $(f|S)_* \pi_1(S)$ is of finite index in $\pi_1(S')$ where $S$ is any component of $\partial M$ and $S'$ is the component of $\partial N$ containing $f(S)$; and

(p.2) the map $\pi_0(\partial M) \to \pi_0(\partial N)$ induced by $f$ is injective.

From now on we will assume that $f$ satisfies (p.1) and (p.2). Construct a commutative diagram

$$
\begin{array}{ccc}
\overline{M} & \xrightarrow{f} & \overline{N} \\
\downarrow \rho & & \downarrow \varrho \\
M & \xrightarrow{f} & N
\end{array}
$$

when $\overline{M}$, $\overline{N}$ are orientable and $\rho$, $\varrho$ are finite covers of the same degree $< 4$. Since $f$ satisfies (p.1), $\overline{f}$ also satisfies (p.1). If $\overline{f}$ does not satisfy (p.2), it follows as above that $\overline{M}$ is a product $I$-bundle. Since $\overline{M}$, $\overline{N}$ are orientable, $\overline{N}$ is also a product $I$-bundle and $\overline{f}$ can be properly deformed into $\partial \overline{N}$. Since $f$ satisfies (p.2), it follows that $M$ is a nontrivial $I$-bundle and that $f_*(\pi_1(M))$ is peripheral in $N$. Hence $f$ can be properly deformed into $\partial N$. Hence either (ii) holds or $\overline{f}$ satisfies (p.1) and (p.2). Thus, we see that either (iv) or (ii) holds or both $f$ and $\overline{f}$ satisfy (p.1) and (p.2). In the later case we choose local orientations for $M$, $N$ and orient $\overline{M}$, $\overline{N}$ accordingly. Since $\overline{f}$ satisfies (p.1), (p.2) and since $\overline{M}$, $\overline{N}$ are orientable, we see that the degree of $\overline{f}$ is nonzero. If $\chi(M) \neq 0$, then $\chi(\overline{M}) \neq 0$ and the degree of $\overline{f}$ has to be $\pm 1$ (see the argument [3, pp. 146–147]). Now Lemmas 3 and 4 show that the degree of $f$ is $\pm 1$ and by the extension of Lemma 3.1 of [2], $f|\partial M$ can be deformed to a homeomorphism. Hence $f$ can be deformed to a homeomorphism as in [3] and (i) holds. It remains to consider the case when $\chi(M) = \chi(\overline{M}) = 0$ and $\deg f \neq \pm 1$. We claim that $\overline{S}$ is compressible in $\overline{M}$. Otherwise, since the degree of $(\overline{f}|\overline{S})_* \pi_1(\overline{S})$ is a proper rank two subgroup of $\pi_1(\overline{f}(\overline{S}))$ and it follows that $\overline{M}$, $\overline{N}$ are nontrivial $I$-bundles. In this case $(\overline{f}|\overline{S})_*$ has to be an isomorphism. This contradiction shows that $\overline{S}$ is compressible in $\overline{M}$ and it follows as in [3] that (iii) holds. This completes the proof of Theorem 1.
REFERENCES

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