BOUNDARY PRESERVING MAPS OF 3-MANIFOLDS

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Abstract. We prove an extension of Waldhausen's theorem [5] conjectured by Hempel in [3].

We prove the following extension of Waldhausen's theorem [5]:

**Theorem 1.** Let $M, N$ be $P^2$-irreducible 3-manifolds. Suppose that $M$ is compact, sufficiently large and $f: (M, \partial M) \to (N, \partial N)$ is a continuous map inducing an injection $f_\#: \pi_1(M) \to \pi_1(N)$. Then, there is a proper homotopy $f_t: (M, \partial M) \to (N, \partial N)$ such that $f_0 = f$ and either

(i) $f_t: M \to N$ is a covering map, or

(ii) $M$ is an $I$-bundle over a closed surface, and $f_1(M) \subset \partial N$, or

(iii) $N$ (hence also $M$) is a solid torus or a solid Klein bottle and $f_1: M \to N$ is a branched covering with branch set a circle, or

(iv) $M$ is a cube with handles and $f_1(M) \subset \partial N$.

If $f|B: B \to C$ is already a covering map, we may assume $f_t|B = f|B$, for all $t$ (where $B$ is any component of $\partial M$ and $C$ the component of $\partial N$ containing $f(B)$).

This theorem is conjectured in [3], where it is proved under additional restrictions. When $M, N$ are orientable and $f_\#$ is an isomorphism, a variant of this result is proved by Evans in [2]. Our argument yields a simple proof of his result too. We refer to [1] and [4] for the concepts 'geometric degree', 'absolute degree', 'orientation-true', etc. The term 'degree of a map $f$' will be used for the twisted degree of $f$ and will be denoted by $\deg f$ as in [4].

**Lemma 2.** Let $M$ be a compact irreducible 3-manifold and let $f: (M, \partial M) \to (N, \partial N)$ be a map into any aspherical 3-manifold such that $f_\#: \pi_1(M) \to \pi_1(N)$ is injective. Let $S$ be a component of $\partial M$, $S'$ the component $\partial N$ containing $f(S)$. If the geometric degree of $(f|S): S \to S'$ is zero, then $M$ is a cube with handles and $f$ is properly homotopic to a map into $\partial N$.

**Remark.** If $(f|S)_\#(\pi_1(S))$ is free subgroup of $\pi_1(S')$, then the geometric degree of $f|S$ is zero.

**Proof.** Since the geometric degree of $(f|S)$ is zero, $(f|S)$ is homotopic to a map of $S$ into $S'-P$ for any $p \in S'$. Hence $(f|S)_\#(\pi_1(S)) = H$ is a finitely generated free subgroup of $\pi_1(S'-p)$. Representing $H$ by the fundamental group of a wedge $X$ of circles, we can write $(f|S)$, up to homotopy, as a composite $S \to X \to S' - p$. 

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Making $g$ transverse to a point of $X$ (≠ base point of $X$) we can find a nontrivial simple loop $l$ in $S$ such that $g(l)$ is null-homotopic in $X$ and hence $f(l)$ is null-homotopic in $S' - p$. Since $f_\ast: \pi_1(M) \to \pi_1(N)$ injective, $l$ bounds a disc $D$ in $M$. Split $M$ along $D$ to obtain $M_1, M_2$ (one of which may be empty) and maps $g_\ast: M_\ast \to N$. Since $f(l)$ is null-homotopic in $S' - p$ and since $N$ is aspherical we may assume that $g_\ast$ maps $(M_\ast, \partial M_\ast)$ to $(N, \partial N)$, $i = 1, 2$. If $S_i$ is the component of $\partial M_i$ obtained from $S$, $g_\ast$ maps $S_i$ into $S' - p$ and therefore the degree of $(g_\ast|S_i)$ is zero. Induction completes the proof of Lemma 2.

The following two lemmas follow easily from [1] and [4]; we sketch the proofs:

**Lemma 3.** Let

$$
\begin{array}{ccc}
(\tilde{M}, \partial \tilde{M}) & \xrightarrow{\tilde{f}} & (\tilde{N}, \partial \tilde{N}) \\
p \downarrow & & \downarrow q \\
(M, \partial M) & \xrightarrow{f} & (N, \partial N)
\end{array}
$$

be a commutative diagram, where $p, q$ are covering projections with the same finite number of sheets and $f_\ast: \pi_1(M) \to \pi_1(N)$ is surjective. Then the geometric degree of $f$ is equal to the geometric degree of $\tilde{f}$.

**Proof.** Let $G(f), A(f)$ and $a(f, 2)$ denote the geometric degree, absolute degree, and the mod 2 degree of $f$ and similarly for $\tilde{f}$. By a theorem of Hopf (see [1]), $G(f) = A(f)$. By Theorem 3.1 of [1], $A(f) = a(f, 2)$ mod 2. From the definition of geometric degree it is immediate that $G(f) > G(\tilde{f})$ and $G(f) = G(\tilde{f})$ mod 2. This together with the above assertions implies that $G(f) = G(\tilde{f})$ if $G(f) < 1$. Hence, it remains to consider the case when $G(f) > 1$. From the classification into three types used in defining absolute degree (see [1, p. 371] and [4, p. 375]), if $f$ is of type II or III, then $G(f) = A(f) < 1$. Since $G(f) > 1$, $f$ has to be of type I and hence it is orientation-true. In this case, by Theorem VIII of [4], $A(f) = |\deg f|$. By Lemma 3.6 of [4],

|\deg f| |\deg p| = |\deg (fp)| = |\deg (q\tilde{f})| = |\deg q| |\deg \tilde{f}|

Since, by assumption $|\deg p| = |\deg q|$, we have $|\deg f| = |\deg \tilde{f}|$ and therefore $G(f) = G(\tilde{f})$ in the last case too.

**Lemma 4.** Let $f: (M, \partial M) \to (N, \partial N)$ be a map inducing isomorphisms in the fundamental groups. If the geometric degree of $f$ is nonzero, then $f_\ast: \pi_1(M) \to \pi_1(N)$ is orientation-true (and the modulus of the degree of $f$ is equal to the geometric degree of $f$).

**Proof.** Since $G(f) \neq 0$ and since $f$ has no kernel, $f$ is of type I and hence is orientation-true. Hence $G(f) = |\deg f|$ by Theorem VIII of [4].

Lemma 2 is what is needed to extend the argument of [3] to the general case; we will quickly sketch the proof along the lines of proof of Theorem 13.6 of [3].
Proof of Theorem 1. Since we are not assuming that \( N \) is compact, the theorem will follow from the case when \( f_* \) is an isomorphism. From now on we will assume that \( f_* \) is an isomorphism. We will show that either (ii) or (iii) or (iv) of Theorem is valid or \( f \) is orientation-true and the degree of \( f \) is \( \pm 1 \). Then by the extension of Lemma 3.1 of [2] to the nonorientable case, Theorem 1 follows.

By Lemma 2, either (iv) is valid or \( (f|S)_*:S \rightarrow S' \) has nonzero geometric degree for each component \( S \) of \( \partial M \); here \( S' \) denotes the component of \( \partial N \) containing \( f(S) \). In the latter case \( (f|S)_*\pi_1(S) \) is a subgroup of finite index in \( \pi_1(S') \). Thus, if two components \( S, T \) of \( \partial M \) map into \( S' \), then \( (f|S)_*\pi_1(S) \) and \( (f|T)_*\pi_1(T) \) intersect in a subgroup of finite index. Now one sees as in [3], that \( M \) is a product \( I \)-bundle and (ii) is valid. Thus, either (iv) or (ii) is valid or, \( f \) satisfies

(p.1) \( (f|S)_*\pi_1(S) \) is of finite index in \( \pi_1(S') \) where \( S \) is any component of \( \partial M \) and \( S' \) is the component of \( \partial N \) containing \( f(S) \); and

(p.2) the map \( \pi_0(\partial M) \rightarrow \pi_0(\partial N) \) induced by \( f \) is injective.

From now on we will assume that \( f \) satisfies (p.1) and (p.2). Construct a commutative diagram

\[
\begin{array}{ccc}
\overline{M} & \xrightarrow{f} & \overline{N} \\
p \downarrow & & \downarrow q \\
M & \xrightarrow{f} & N
\end{array}
\]

when \( \overline{M}, \overline{N} \) are orientable and \( p, q \) are finite covers of the same degree \( < 4 \). Since \( f \) satisfies (p.1), \( \overline{f} \) also satisfies (p.1). If \( \overline{f} \) does not satisfy (p.2), it follows as above that \( \overline{M} \) is a product \( I \)-bundle. Since \( \overline{M}, \overline{N} \) are orientable, \( \overline{N} \) is also a product \( I \)-bundle and \( \overline{f} \) can be properly deformed into \( \partial \overline{N} \). Since \( f \) satisfies (p.2), it follows that \( M \) is a nontrivial \( I \)-bundle and that \( f_* (\pi_1(M)) \) is peripheral in \( N \). Hence \( f \) can be properly deformed into \( \partial N \). Hence either (ii) holds or \( \overline{f} \) satisfies (p.1) and (p.2).

Thus, we see that either (iv) or (ii) holds or both \( f \) and \( \overline{f} \) satisfy (p.1) and (p.2). In the later case we choose local orientations for \( M, N \) and orient \( \overline{M}, \overline{N} \) accordingly. Since \( \overline{f} \) satisfies (p.1), (p.2) and since \( \overline{M}, \overline{N} \) are orientable, we see that the degree of \( \overline{f} \) is nonzero. If \( \chi(M) \neq 0 \), then \( \chi(\overline{M}) \neq 0 \) and the degree of \( \overline{f} \) has to be \( \pm 1 \) (see the argument [3, pp. 146–147]). Now Lemmas 3 and 4 show that the degree of \( f \) is \( \pm 1 \) and by the extension of Lemma 3.1 of [2], \( f_*|\partial M \) can be deformed to a homeomorphism. Hence \( f \) can be deformed to a homeomorphism as in [3] and (i) holds. It remains to consider the case when \( \chi(M) = \chi(\overline{M}) = 0 \) and \( \deg f \neq \pm 1 \). We claim that \( \overline{S} \) is compressible in \( \overline{M} \). Otherwise, since the degree of \( (\overline{f}|\overline{S})_* \) \( \neq \pm 1 \), \( (\overline{f}|\overline{S})_*\pi_1(\overline{S}) \) is a proper rank two subgroup of \( \pi_1(\overline{f}(\overline{S})) \) and it follows that \( \overline{M}, \overline{N} \) are nontrivial \( I \)-bundles. In this case \( (\overline{f}|\overline{S})_* \) has to be an isomorphism. This contradiction shows that \( \overline{S} \) is compressible in \( \overline{M} \) and it follows as in [3] that (iii) holds. This completes the proof of Theorem 1.
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