THE CLOSED SOCLE OF AN AZUMAYA ALGEBRA

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Abstract. If $R$ is a Noetherian ring and $A$ is an Azumaya algebra over $R$ then an ideal $H(A)$ in $R$, called the closed socle of $A$, is defined and it is shown that $H(A)$ is independent of the representative $A$ in the Brauer group of $R$. When $R$ is a domain, the behavior of $H(A)$ under localization and passage to the quotient field is studied, and $H(A)$ is calculated when $R$ is the affine ring of a real curve.

Let $R$ denote a Noetherian, integrally closed domain and $A$ an Azumaya (central separable) algebra over $R$. In [6], D. Haile associated to $A$ an ideal in $R$ which he called the closed socle of $A$ and which we denote $H(A)$. In this note we show how to define the closed socle of an Azumaya algebra over any commutative Noetherian ring, give simplified proofs of results slightly more general than those in [6], and calculate the closed socle of an Azumaya algebra over the affine ring of a real curve. More specifically, if $R$ is a commutative Noetherian ring and $A$ is an Azumaya algebra over $R$ then $H(A)$ is defined and is independent of the choice of representative $A$ in its class in the Brauer group $B(R)$ of $R$. If $R$ is a local Noetherian domain with quotient field $F$ and maximal ideal $m$ and $2 = A \otimes F$, then $\text{Index}(\Sigma) > \text{Index}(A/mA)$ when $H(A) = R$. Thus, if $\Sigma = M_\omega(F)$ then $H(A) = R$ if and only if $A = M_\omega(R)$. Also, if $A/mA$ is a division algebra then $A$ is a maximal order in a division algebra over $F$ when $H(A) = R$. A localization result is proved and consequences of these results for Noetherian domains are derived. If $R$ is the affine ring of a real curve $X$ and $A$ is an Azumaya algebra over $R$ then $H(A) \subseteq \cap P_x$ where $P_x$ is the prime ideal in $R$ corresponding to a point $x \in X$ and $x$ runs over the singular points $x \in X$ which are isolated in the strong topology and for which $A/P_xA$ is not in the trivial class of $B(R/P_x)$. Throughout all unexplained terminology and notation is as in [4], and $\otimes$ always means $\otimes_R$.

1. We begin by extending the definition of the closed socle given in [6]. Let $R$ be a domain with quotient field $F$, let $A$ be an Azumaya algebra over $R$, and let $\Sigma = A \otimes F = A \cdot F$. A left ideal $L$ in $A$ is pure over $R$ in case $ra \in L$ for $0 \neq r \in R$ and $a \in A$ implies $a \in L$ ([2, p.199]). It is easy to show that there is a one-to-one order preserving correspondence between the $R$-pure left ideals $L$ in $A$ and the left ideals $L^1$ in $\Sigma$ by $L \rightarrow L \cdot F$ and $L^1 \rightarrow L^1 \cap A$. It follows that minimal pure left ideals exist in $A$. Let $I$ be the sum of the minimal $R$-pure left ideals in $A$, then the closed socle of $A$ is defined to be $I \cap R$ and is denoted $H(A)$. Observe that the set $I$ in $A$ is actually a two-sided ideal in $A$. It is clear that $I$ is a left ideal in $A$, and for $a \in A$ and minimal pure left ideal $L$ of $A$ either $La = 0$ or

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299

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(La)\(F = (LF)a\) is a minimal left ideal of \(\Sigma\), since \(La \subseteq (LF)a \cap A\) and \((LF)a \cap A\) is a two-sided ideal in \(A\). By the one-to-one correspondence between two-sided ideals of \(A\) and \(R\) ([4, 3.7, p. 54]) it follows that \(I = A \cdot H(A)\). This corresponds to the construction of the closed socle for normal domains given in [6].

Now assume \(R\) is a reduced Noetherian ring. Then \((0) = \bigcap \mathfrak{p}_i\), where the \(\mathfrak{p}_i\) are a uniquely determined set of prime ideals so that the intersection is irredundant. If \(A\) is an Azumaya algebra over \(R\) let \(A_i = R/\mathfrak{p}_i \otimes A\), then \(A_i\) is an Azumaya algebra over the domain \(R/\mathfrak{p}_i\). From the above \(H(A_i)\) is an ideal in \(R/\mathfrak{p}_i\) whose natural inverse image is an ideal \(I_i\) in \(R\). We define the closed socle \(H(A)\) of \(A\) to be \(\bigcap \mathfrak{p}_i I_i\). If \(R\) is just a commutative Noetherian ring let \(N\) be the nil radical of \(R\), and let \(A\) be an Azumaya algebra over \(R\). Above we defined \(H(A/NA)\) in \(R/N\). We define the closed socle \(H(A)\) of \(A\) to be the natural inverse image of \(H(A/NA)\) in \(R\). The calculation of the closed socle of an Azumaya algebra over the affine ring of a real curve given later illustrates the naturalness of this definition.

**Lemma 1.** Let \(R\) be a domain and \(P\) a prime ideal in \(R\). Let \(A\) be an Azumaya algebra over \(R\), then \(H(R_P \otimes A) = R_P \otimes H(A)\) where \(R_P\) denotes the localization of \(R\) at \(P\).

**Proof.** One can check that there is a one-to-one correspondence between the minimal pure left ideals \(L\) of \(A\) and the minimal pure left ideals \(L^1\) of \(R_P \otimes A = A R_P\) by \(L \rightarrow LR_P\) and \(L^1 \rightarrow A \cap L^1\). Let \(I\) be the sum of the minimal pure left ideals of \(A\) and \(I^1\) the sum of the minimal pure left ideals in \(A^1\). Then \(I \cdot R_P = \Sigma LR_P\) where \(L\) runs through the minimal pure left ideals in \(A\). Thus \(IR_P = I^1\).

Now one can check that \(H(A)R_P = H(R_P \otimes A)\).

We note here that if \(S\) is a commutative \(R\)-algebra and \(A\) is an Azumaya \(R\)-algebra it may not be the case that \(H(S \otimes A) = S \otimes H(A)\). For example let \(R\) denote the field of real numbers, let \(R = R[x, y]/(x^2 + y^2)\) and let \(S\) be the integral closure of \(R\). We later show that if \(A\) is the Azumaya algebra generated by elements \(i, j\) subject to \(i^2 = j^2 = -1\) and \(ij = -ji\) and \(P\) is the maximal ideal in \(R\) generated by \(\{x, y\}\) then \(H(A) \subseteq P\); but \(H(S \otimes A) = S\) so \(S \otimes H(A) \neq H(S \otimes A)\).

**Lemma 2.** Let \(A\) and \(A^1\) be two Azumaya algebras in the same class of the Brauer group of the Noetherian ring \(R\), then \(H(A) = H(A^1)\).

**Proof.** Assume \(R\) is a local domain. Then there is an Azumaya \(R\)-algebra \(D\) with no idempotents other than 0 and 1 and positive integers \(n\) and \(k\) with \(A \cong M_n(D), A^1 = M_k(D)\) by Corollary 1 of [3]. In this case it suffices to show \(H(D) = H(M_n(D))\). Let \(F\) be the quotient field of \(R\) and \(\Sigma = F \otimes D = F \cdot D\). Then \(F \otimes M_n(D) = M_n(\Sigma)\). Let \(e_{ij}\) be the matrix in \(M_n(\Sigma)\) with 1 in the \(i, j\) entry and 0 elsewhere. Let \(L\) be a minimal pure left ideal in \(D\) so \(L^1 = \Sigma L\) is a minimal left ideal in \(\Sigma\). Then \(M_n(L) = \bigoplus \epsilon_{\mathfrak{m}_i} M_n(L)e_{ii}\) and \(M_n(\Sigma) \cdot M_n(L) = M_n(\Sigma) \cdot L = \bigoplus \epsilon_{\mathfrak{m}_i} M_n(\Sigma L)e_{ii}\). Moreover, since \(L\) is \(R\)-pure in \(\Sigma\), \(M_n(L)e_{ii}\) is \(R\)-pure in \(M_n(\Sigma)\) so \(M_n(\Sigma L)e_{ii} \cap M_n(D) = M_n(L)e_{ii}\). Finally, \(M_n(\Sigma L)e_{ii}\) is a minimal left ideal in \(M_n(\Sigma)\) since \(\Sigma \cdot L\) is a minimal left ideal in \(\Sigma\). Thus
\[ M_n(L) = M_n(\Sigma L) \cap M_n(D) = \left( \bigoplus_{i=1}^{n} M_n(\Sigma L)e_{ii} \right) \cap M_n(D) \]
\[ \supseteq \bigoplus_{i=1}^{n} M_n(\Sigma L)e_{ii} \cap M_n(D) = \bigoplus_{i=1}^{n} M_n(L)e_{ii} = M_n(L). \]

Thus \( M_n(L) \) is contained in the sum of the minimal pure left ideals in \( M_n(D) \) so \( H(D) \subseteq H(M_n(D)) \). For the reverse inclusion let \( I \) be the sum of the minimal pure left ideals \( L \) of \( D \). Let \( J' \) be the sum of minimal pure left ideals in \( M_n(D) \). By 3.5, p. 22 in [4] we know \( J' = M_n(J) \) for some two-sided ideal \( J \) in \( D \). We have already shown \( I \subseteq J \). Let \( (x_{ij}) \) be an \( n \times n \) matrix in \( M_n(D) \) with \( (x_{ij}) \notin M_n(I) \) yet \( (x_{ij}) \in L_1 \) for some minimal pure left ideal \( L_1 \) of \( M_n(D) \). If \( x_{ij} \notin I \) then \((e_{ij})(x_{ij})(e_{ij})\) is contained in \( M_n(D) \) and in the minimal left ideal \( L'e_{ij} \). Thus if \( x_{ij} \) is the matrix whose \( ij \)th entry is \( x_{ij} \) and all others are 0 then \( x_{ij} \notin L'e_{ij} \cap M_n(D) \) is in a minimal pure left ideal in \( M_n(D) \), so \( M_n(\Sigma)X_{ij} \) is a minimal left ideal in \( M_n(\Sigma) \). Thus \( \Sigma x_{ij} \) must be a minimal left ideal in \( \Sigma \). Thus \( x_{ij} \in \Sigma x_{ij} \cap D \subseteq I \) which contradicts the choice of \( x_{ij} \).

Now let \( R \) be any commutative Noetherian ring with nil radical \( N \). If \( A \) is equivalent to \( A^1 \) in \( B(R) \) then \( A/N(A) \) is equivalent to \( A^1/N^1 \) in \( B(R/N) \) so it suffices to show \( H(A/N(A)) = H(A^1/N^1) \). Similarly, we can assume \( R \) is a domain for if we write (0) as the irredundant intersection of a unique set \( \{P_i\}_{i=1}^{m} \) of prime ideals and if \( H(A/P_iA) = H(A^1/P_iA^1) \) for all \( i \) then \( H(A) = H(A^1) \). If \( R \) is a Noetherian domain and \( P \) is any prime ideal in \( R \) we know \( R_P \otimes H(A) = R_P \otimes H(A^1) \) by the first part of the proof since \( R_P \otimes A \) is equivalent to \( R_P \otimes A^1 \) in \( B(R_P) \). Thus if \( H(A) = \bigcap_{i=1}^{m} P_i^\infty \) and \( H(A^1) = \bigcap_{i=1}^{m} P_i^\infty \) where \( \{Q_i\} \) is an irredundant collection of prime ideals and \( m_i > 0 \) then \( m_i = n_i \) for all \( i \) so we always have \( H(A) = H(A^1) \).

If \( A \) is an Azumaya algebra over a field \( F \) then the index of \( A \) is the square root of the dimension of the division algebra part of \( A \) over \( F \). The next result is a generalization (with an easier proof) of Theorem 4.6 of [6].

**Theorem 1.** Let \( R \) be a local Noetherian domain with maximal ideal \( m \) and quotient field \( F \). Let \( A \) be an Azumaya algebra over \( R \) and \( \Sigma = F \otimes A \). If \( H(A) = R \) then Index \( \Sigma > \) Index \( A/mA \). Moreover, if Index \( \Sigma = \) Index \( A/mA \) then \( A = M_n(B) \) where \( B \) is an Azumaya \( R \)-algebra such that \( F \otimes B \) is a division algebra.

**Proof.** Assume \( H(A) = R \). Then there is a minimal pure left ideal \( L \) of \( A \) with \( L \not\subseteq mA \). Let \( A_0 = \Sigma La \) where the sum runs over those \( a \in A \) such that \( La \not\subseteq mA \). Then \( A_0 \) is an \( R \)-submodule of \( A \) and \( (A_0 + mA)/mA \) is a two-sided ideal in \( A/mA \). Since \( A/mA \) is simple, \( (A_0 + mA)/mA = A/mA \), so by Nakayama's lemma, \( A_0 = A \). Therefore \( A = \Sigma La \) where the sum is taken over finitely many \( a \in A \) such that \( La \not\subseteq mA \). Let \( La = F \cdot La \). Then \( \Sigma = \bigoplus_{a=1}^{n} \Sigma La \). Thus \( A \supseteq \bigoplus_{a=1}^{n} \Sigma La \). Note that \( (A_1 + mA)/mA = (La_1 + mA)/mA \otimes (La_1 + mA)/mA \) and each summand on the right is a nonzero ideal in \( A/mA \). This proves Index \( \Sigma > \) Index \( A/mA \). If Index \( \Sigma = \) Index \( A/mA \) then \( (A_1 + mA)/mA = A/mA \) so by
Nakayama's lemma $A_1 = A$. Let $B^0 = \text{Hom}_A(La_1, La_1)$. Then $B$ is an Azumaya algebra over $R$ and $A = \text{Hom}_R(La_1, La_1) = M_n(B)$, with $B$ a maximal order in the division algebra component of $\Sigma$.

**Corollary 1.** Let $R$ be a local Noetherian domain with quotient field $F$, and let $A$ be an Azumaya $R$-algebra. If $F \otimes A \cong M_n(F)$ then $A \cong M_n(R)$ if and only if $H(A) = R$.

**Proof.** If $A \cong M_n(R)$ then it is easy to see that $H(A) = R$ (this also follows from Lemma 2). Conversely, Index $F \otimes A = 1 > \text{Index } A/mA$ so the inequality is an equality and the result follows from Theorem 1.

**Corollary 2.** Let $R$ be a Noetherian domain with field of quotients $F$. Suppose $A$ is an Azumaya $R$-algebra with $F \otimes A \cong M_n(F)$. Let $P$ be a prime ideal in $R$, then $R_P \otimes A \cong M_n(R_P)$ if and only if $H(A) \not\subseteq P$.

**Proof.** Combine Lemma 1 and Corollary 1.

**Corollary 3.** Let $R$ and $A$ be as in Theorem 1. If $A/mA$ is a division algebra and $H(A) = R$ then $F \otimes A$ is a division algebra.

**Proof.** By Theorem 1, Index $F \otimes A > \text{Index } A/mA$ since $[\text{Index } A/mA]^2 = \text{Rank}_{R/m}(A/mA) = \text{Rank}_R(A) = \text{Rank}_F(F \otimes A) > [\text{Index } F \otimes A]^2$. Thus $[\text{Index } F \otimes A]^2 = \text{Rank}_F(A)$ and $F \otimes A$ is a division algebra.

**Theorem 2.** Let $R$ be the affine ring of a real curve $X$. For each closed point $x \in X$ let $P_x$ be the corresponding maximal ideal in $R$. Let $A$ be an Azumaya $R$-algebra. Then the longest product of ideals $P_x$ containing $H(A)$ has factors $P_x$ satisfying all of the following.

1. $x$ is a real singular point on $X$.
2. $x$ is isolated in the strong topology on the real points of the irreducible component $X_i$ of $X$ containing $x$.
3. $A/P_xA$ does not represent the trivial class in $B(R/P_x)$.

**Proof.** For each $x \in X$ let $R(x) = R/P_x$ and $A(x) = A/P_xA$. Since $X$ is a real curve $R(x)$ is either the real or complex numbers. If $R(x)$ is the real number $\mathbb{R}$ then either $A(x)$ is a matrix algebra over $\mathbb{R}$ or $A(x)$ is a matrix algebra over the division algebra of real quaternions. In [5] the Azumaya algebras $A$ over $R$ are characterized as continuously parameterized systems of Azumaya algebras $A(x)$ over the real points $x \in X$ with the strong topology. If $X$ is irreducible then $R$ is a domain and the quotient field $F$ of $R$ is the field of rational functions on $X$. By Remark 3.4 of [5] the real points $x$ for which $A(x)$ is a matrix algebra over the quaternions, yet $F \otimes_{R(x)} A(x)$ is a matrix algebra over $F$, are precisely the real singular points $x \in X$ which are isolated in the strong topology and for which $A(x)$ is a matrix algebra over the quaternions. Combining Theorem 1 and Lemma 1 we see that $A(x)$ is a matrix algebra over the quaternions yet $F \otimes A(x)$ is a matrix algebra over $F$ if and only if $P_x$ contains $H(A)$. If $X$ is reducible we can assume $R$ is reduced. Let $(0) = \cap P_i$, where $\{P_i\}$ is an irredundant set of prime ideals of $R$. The rings $R/P_i$ are the affine rings of the irreducible components $X_i$ of $X$. If
$A_i = R/P_i \otimes A$ then $H(A_i)$ is contained in the product of the maximal ideals which correspond to the real singular points $x_i$ on $X_i$ which are isolated in the strong topology on $X_i$ and for which $A(x)$ is a matrix algebra over the quaternion algebra. This completes the proof of the theorem.

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