MODULES WITH ARTINIAN PRIME FACTORS

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Abstract. An \( R \)-module \( M \) has Artinian prime factors if \( M/PM \) is an Artinian module for each prime ideal \( P \) of \( R \). For commutative rings \( R \) it is shown that Noetherian modules with Artinian prime factors are Artinian. If \( R \) is either commutative or a von Neumann regular \( V \)-ring then the endomorphism ring of a module with Artinian prime factors is a strongly \( \pi \)-regular ring.

A ring \( R \) with 1 is left \( \pi \)-regular if for each \( a \in R \) there is an integer \( n > 1 \) and \( b \in R \) such that \( a^n = a^n + lb \). Right \( \pi \)-regular is defined in the obvious way, however a recent result of F. Dischinger [5] asserts the equivalence of the two concepts. A ring \( R \) is \( \pi \)-regular if for any \( a \in R \) there is an integer \( n > 1 \) and \( b \in R \) such that \( a^n = a^n ba^n \). Any left \( \pi \)-regular ring is \( \pi \)-regular but not conversely. Because of this, we say that \( R \) is strongly \( \pi \)-regular if it is left (or right) \( \pi \)-regular.

In [2, Theorem 2.5] it was established that if \( R \) is a (von Neumann) regular ring whose primitive factor rings are Artinian and if \( M \) is a finitely generated \( R \)-module then the endomorphism ring \( \text{End}_R(M) \) of \( M \) is a strongly \( \pi \)-regular ring. Curiously enough, the same is not true for finitely generated modules over strongly \( \pi \)-regular rings, as Example 3.1 of [2] shows. Obviously, it also fails for arbitrary regular rings. These observations lead one to consider conditions on finitely generated modules which ensure that the endomorphism ring is strongly \( \pi \)-regular. A natural one seems to be that of having Artinian prime factors. In fact, we establish that such modules have strongly \( \pi \)-regular endomorphism ring whenever the base ring is either commutative or a regular \( V \)-ring.

Consider a finitely generated module \( M \) over a ring \( R \). If \( R \) is commutative then \( R \) is \( \pi \)-regular if and only if its prime ideals are maximal [11]. Accordingly, when \( R \) is commutative and \( \pi \)-regular, \( M/PM \) is an Artinian module for all primes \( P \). This observation serves as a starting point.

Theorem 1. Suppose \( R \) is a commutative ring. For a finitely generated \( R \)-module \( M \) the following conditions are equivalent.

(a) \( M \) has Artinian prime factors.
(b) \( S = \text{End}_R(M) \) is a strongly \( \pi \)-regular ring.
(c) \( R/\text{Ann}_R(M) \) is a \( \pi \)-regular ring.

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Proof. (b) $\Rightarrow$ (c): Since $M$ is finitely generated, $S$ satisfies a polynomial identity. Hence each prime ideal of $S$ is a maximal ideal (see e.g. [1]). Now $R' = R/\text{Ann}_R(M)$ embeds in the center of $S$ and $S$ is an integral extension of $R'$ [10]. It follows that prime ideals of $R'$ are maximal ideals [3] and hence $R'$ is $\pi$-regular.

(c) $\Rightarrow$ (a): This is clear.

(a) $\Rightarrow$ (b): By [2, Proposition 2.3], $S$ is strongly $\pi$-regular if and only if for each $\alpha \in S$ there is an integer $t > 1$ such that $M = \text{Ker} \alpha^t \oplus \mathcal{M} \alpha^t$. Thus let $\alpha \in S$. Our first step is to show that there is an integer $t > 1$ such that $\mathcal{M} \alpha^t = \mathcal{M} \alpha^{t+1}$. Assume that no such integer exists. Because $M$ is finitely generated, there is an ideal $P$ of $R$ which is maximal among those ideals $I$ of $R$ having the property that $\mathcal{M} \alpha^k \subseteq \mathcal{M} \alpha^{k+1} + IM$ for all integers $k > 1$. We claim that $P$ is a prime ideal of $R$. Thus suppose $A$ and $B$ are ideals of $R$ properly containing $P$ and such that $AB \subseteq P$. We then have integers $m, n$ such that $\mathcal{M} \alpha^m \subseteq \mathcal{M} \alpha^{m+1} + AM$ and $\mathcal{M} \alpha^n \subseteq \mathcal{M} \alpha^{n+1} + BM$. The second of these inclusions gives us $AM \mathcal{M} \alpha^n \subseteq AM \mathcal{M} \alpha^{n+1} + PM$, since $AB \subseteq P$. Hence $AM \mathcal{M} \alpha^{n+1} \subseteq AM \mathcal{M} \alpha^{n+2} + PM$, giving $AM \mathcal{M} \alpha^n \subseteq AM \mathcal{M} \alpha^{n+2} + PM$. Continuing we arrive at $AM \mathcal{M} \alpha^n \subseteq AM \mathcal{M} \alpha^{n+m+1} + PM$ and therefore $AM \mathcal{M} \alpha^n \subseteq AM \mathcal{M} \alpha^{n+m+1} + PM$. Using this we then get $AM \mathcal{M} \alpha^{n+m} = (AM \mathcal{M}) \alpha^n \subseteq (AM \mathcal{M}) \alpha^{n+1} + AM \mathcal{M} \alpha^n \subseteq AM \mathcal{M} \alpha^{n+m+1} + PM$, which contradicts the choice of $P$. Thus $P$ is a prime ideal as claimed. By assumption, $M/PM$ is an Artinian module. But then the sequence of submodules $\mathcal{M} \alpha \supseteq \mathcal{M} \alpha^2 \supseteq \cdots$ must terminate modulo $PM$, providing the desired contradiction. This shows then that $\mathcal{M} \alpha^t = \mathcal{M} \alpha^{t+1}$ for some integer $t > 1$. Now $\alpha$ is an onto endomorphism of the finitely generated $R$-module $\mathcal{M} \alpha^t$, and so $\alpha$ is 1-1 on $\mathcal{M} \alpha^t$ since $R$ is commutative [12]. Then $\ker \alpha \cap \mathcal{M} \alpha^t = 0$ implies that $\ker \alpha^t = \ker \alpha^{t+1}$. It now follows easily that $M = \ker \alpha^t \oplus \mathcal{M} \alpha^t$, completing the proof.

Examination of the proof of the implication (a) $\Rightarrow$ (b) shows that commutativity was used only to ensure that onto endomorphisms are 1-1. It has been shown in [2, Theorem 2.2] that rings integral over their center and satisfying a polynomial identity have the property that onto endomorphisms of finitely generated modules are 1-1, a property which left Noetherian rings also have. Thus we are able to state the following.

Theorem 2. Assume $R$ is either a PI-ring integral over its center or a left Noetherian ring. If $M$ is a finitely generated left $R$-module having Artinian prime factor then $\text{End}_R(M)$ is a strongly $\pi$-regular ring.

In view of this theorem one might ask if any Noetherian $R$-module with Artinian prime factors is Artinian. The answer is no, in general. An example in [9, p. 66] provides us with a perfect ring $D$ having a Noetherian non-Artinian module. Since $D/P$ is simple Artinian for any prime ideal $P$, such a module must have Artinian prime factors. Before showing that the answer is affirmative when $R$ is commutative, we note that over a semiprimary ring, all Noetherian modules are Artinian, so the answer is (trivially) yes in this case.

Theorem 3. If $R$ is a commutative ring and $M$ is a Noetherian $R$-module with Artinian prime factors then $M$ is Artinian.
Proof. By Theorem 1, \( R' = R/\text{Ann}_R(M) \) is a \( \pi \)-regular ring. Since \( M \) is a faithful finitely generated \( R' \)-module, \( R' \) is isomorphic to a submodule of a finite direct sum of copies of \( M \). Hence \( R' \) is a Noetherian module. Any Noetherian \( \pi \)-regular ring is Artinian so \( R' \) is Artinian. But then \( M \), being a finitely generated \( R' \)-module, must be Artinian.

While it is false in general that Noetherian modules with Artinian prime factors are Artinian, the following is true.

Theorem 4. If \( M \) is a Noetherian \( R \)-module with Artinian prime factors then \( S = \text{End}_R(M) \) is semiprimary.

Proof. The proof of (a) \( \Rightarrow \) (b) shows that \( S \) is a strongly \( \pi \)-regular ring. Thus each nonnil one sided ideal contains a nonzero idempotent. It follows that \( J(S) \), the Jacobson radical of \( S \), is a nil ideal. By a theorem of L. Small (see [6, Theorem 2.1]), nil subrings of \( S \) are nilpotent, so that \( J(S) \) is a nilpotent ideal of \( S \). Now orthogonal idempotents of \( S/J(S) \) can be lifted to \( S \). However \( M \) is Noetherian so \( S \) can have no infinite set of orthogonal idempotents. It follows then that \( S/J(S) \) is a semisimple Artinian ring.

This theorem generalizes the well-known fact that the endomorphism ring of a Noetherian Artinian module is semiprimary.

We now turn to a result which covers [2, Theorem 2.3]. Recall that a (left) \( V \)-ring is a ring all of whose simple left modules are injective. For the salient features of \( V \)-rings we refer the reader to [4, Chapter 5].

Theorem 5. Assume \( R \) is a regular \( V \)-ring. If \( M \) is a finitely generated \( R \)-module with Artinian prime factors then \( S = \text{End}_R(M) \) is a strongly \( \pi \)-regular ring.

Proof. Let \( \alpha \in S \). As in the proof of Theorem 1, there is an integer \( t > 1 \) such that \( Ma^t = Ma^{t+1} \). Suppose \( x \in \text{Ker} \alpha^{t+1} \); if \( u = x\alpha^t \neq 0 \), then there is an ideal \( P \) of \( R \) maximal among those ideals \( I \) of \( R \) for which \( u \notin IM \). Hence \( u \in AM \) for all ideals \( A \) of \( R \) properly containing \( P \). If \( P \) is not a prime ideal then there are ideals \( A \) and \( B \) of \( R \) properly containing \( P \) for which \( AB \subseteq P \). Then \( u \in BM \) so that \( Au \subseteq ABM \subseteq PM \). Since we also have \( u \in AM \) we can write \( u = \Sigma a_i m_i \) where \( a_i \in A \), \( m_i \in M \). Because \( R \) is a regular ring there is an idempotent \( e \in A \) such that \( ea_i = a_i \) for each \( i \). But then \( u = eu \in Au \subseteq PM \), a contradiction. Thus \( P \) must be a prime ideal and the module \( M/PM \) is Artinian. Because \( R \) is a \( V \)-ring and \( M/PM \) has finitely generated essential socle, we infer that \( M/PM \) is completely reducible and hence Noetherian. Then \( \alpha \) induces \( \beta \in \text{End}_R(M/PM) \) and \( (M/PM)\beta^t = (M/PM)\beta^{t+1} \) and this yields \( \text{Ker} \beta^t = \text{Ker} \beta^{t+1} \). But then \( x\alpha^t \in PM \), which is the desired contradiction. It now follows that \( \text{Ker} \alpha^t = \text{Ker} \alpha^{t+1} \), \( M = Ma^t \oplus \text{Ker} \alpha^t \), and so \( S \) is strongly \( \pi \)-regular.

Corollary 5 [2, Theorem 2.3]. If \( R \) is a regular ring whose primitive factor rings are Artinian then \( \text{End}_R(M) \) is strongly \( \pi \)-regular for any finitely generated \( R \)-module \( M \).
Proof. It is enough to note that (i) \( R \) is a \( V \)-ring, and (ii) prime factor rings of \( R \) are Artinian. That (i) holds follows from [4, Corollary 5.13] while [8, Theorem 3, p. 239] guarantees (ii).

It is straightforward to see that a finitely generated projective Artinian module over a semiprime ring is completely reducible. Thus the proof of Theorem 4 can be used to prove the next result.

**Theorem 6.** Let \( R \) be a regular ring and \( M \) a finitely generated \( R \)-module. If \( M/PM \) is a projective Artinian \( R/P \)-module for each prime ideal \( P \) of \( R \) then \( \text{End}_{\pi}(M) \) is strongly \( \pi \)-regular.

In the first version of this article we asked whether or not a finitely generated Artinian module over a regular ring is Noetherian. An affirmative answer would then imply the statement,

over any regular ring, finitely generated modules with Artinian prime factors have a strongly \( \pi \)-regular endomorphism ring. (\( * \))

Recently, K. Goodearl has constructed examples of cyclic Artinian non-Noetherian modules as well as Noetherian non-Artinian modules over regular rings [7]. Thus our original question has a negative answer. However the validity of (\( * \)) still remains open, and would be true should the following question have a positive response. If \( M \) is a finitely generated Artinian module over a regular ring, is every onto endomorphism of \( M \) also 1-1?

**References**


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