LINEAR CONVOLUTION INTEGRAL EQUATIONS WITH ASYMPTOTICALLY ALMOST PERIODIC SOLUTIONS

G. S. JORDAN, W. R. MADYCH AND R. L. WHEELER

Abstract. Let \( \mu \) be a bounded Borel measure and \( f \) be asymptotically almost periodic. Conditions are found which ensure that certain bounded solutions of the linear convolution integral equation \( g \ast \mu = f \) are asymptotically almost periodic. This result is also extended to the case where the measure \( \mu \) is replaced by a tempered distribution \( \tau \) for which convolution with bounded functions makes sense.

1. Classical convolution equations. Recently Fink and Madych [7] studied the asymptotic behavior as \( t \to \infty \) of certain bounded solutions of the linear integral equation

\[
\int_{-\infty}^{\infty} g(t-s) \, d\mu(s) = f(t), \quad -\infty < t < \infty,
\]

where \( \mu \) is a bounded Borel measure on \( \mathbb{R} = (-\infty, \infty) \), and \( f(t) \) belongs to \( L^\infty = L^\infty(\mathbb{R}) \) with \( f(t) \to 0 \) as \( t \to \infty \). In this paper we show that the main result of [7, Theorem 3] may be extended in two directions. First, in this section (see Theorem 1) we show that Theorem 3 of [7] holds for more general forcing functions \( f \) than are considered there. Then, in §2 we show that the conclusion of Theorem 1 holds if the measure \( \mu \) is replaced by a tempered distribution \( \tau \) for which convolution with bounded functions makes sense.

We will use the notation of [7]. In particular, a function \( g \) in \( L^\infty \) is said to satisfy the tauberian condition \( T \) if

\[
\lim_{t \to \infty, s \to 0} |g(t+s) - g(t)| = 0. \tag{T}
\]

If \( m \) is a positive integer, we say that \( f \in L^\infty \) satisfies property \( M(f, m) \) if

\[
\lim_{t \to \infty} f(t) = 0 \quad \text{and} \quad \int_0^{\infty} t^{-m} |f(t)| \, dt < \infty.
\]

By \( M(f, \infty) \) we mean that \( M(f, m) \) holds for each positive integer \( m \). If \( f \) is an almost periodic function \( (f \in \text{a.p.}) \), \( \exp f \) denotes the set of exponents of \( f \) (see [6, Chapter 3]).

Also, \( \mathfrak{M} \) denotes the space of bounded Borel measures. If \( \mu \in \mathfrak{M} \), let \( \hat{\mu}(\xi) = \int_{\infty}^{-\infty} e^{-it\xi} \, d\mu(t) \) be the Fourier transform of \( \mu \) and \( \Lambda(\mu) = \{ \xi \in \mathbb{R} : \hat{\mu}(\xi) = 0 \} \) be the set of zeros of \( \hat{\mu} \). Finally, we say that \( \mu \in \mathfrak{M} \) satisfies property \( M \Lambda(\mu) \) if for every \( \xi_j \in \Lambda(\mu) \) there exists a positive integer \( m_j \) (necessarily unique) such that the function \( \hat{\delta}_j \) defined by \( \hat{\delta}_j(\xi) = (\xi - \xi_j)^{-m_j} \hat{\mu}(\xi) \) is the Fourier transform of a measure

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Theorem 1. Suppose \( g \) is a bounded solution of equation (1), where

(i) \( \Lambda(\mu) \) has no finite accumulation point,
(ii) \( \mu \) satisfies property \( M(\mu) \),
(iii) \( f = f_1 + f_2 \) where \( f_1 \in \text{a.p.} \) and \( f_2 \) satisfies property \( M(f, m) \) for \( m = \sup\{m_j\} \) where the numbers \( m_j \) are defined in the previous paragraph,
(iv) \( g \) satisfies condition \( T \).

Then \( g = g_1 + g_2 \) where \( g_1 \in \text{a.p.} \) with \( \exp g_1 \subseteq \Lambda(\mu) \cup \exp f_1 \), and \( \lim_{t \to \infty} g_2(t) = 0 \).

In the case where \( f_1(t) = 0 \), Theorem 1 is Theorem 3 of [7]. (More restricted versions of [7, Theorem 3] where, in particular, \( \Lambda(\mu) \) is finite are obtained in Jordan and Wheeler [8, Theorem 1], and in Levin and Shea's three part paper [9, Theorem 5c].) On the other hand, in the case where \( f = f_1 \) is a.p., a related result which guarantees that bounded, uniformly continuous solutions of (1) (with \( \mu \) absolutely continuous) are a.p. was obtained by Lewitan [10]. (See, also, Doss [4] and Lemma 2 below.) Finally, the nature of bounded, uniformly continuous solutions of the homogeneous form of equation (1), i.e., with \( f(t) = 0 \), was investigated by Beurling in [2], [1], and now constitutes one aspect of spectral analysis; see [9, §8] for a brief discussion which includes references to appropriate literature.

Our proof of Theorem 1 consists of reducing the problem to considering separately the case where \( f(t) = f_2(t) \to 0 \) as \( t \to \infty \), and the case where \( f = f_1 \) is a.p.

Lemma 1. Suppose \( g \) is a bounded solution of equation (1) which satisfies the tauberian condition \( T \), and \( f = f_1 + f_2 \) where \( f_1 \in \text{a.p.} \) and \( \lim_{t \to \infty} f_2(t) = 0 \). Then there exists a function \( h \) which is bounded, uniformly continuous on \( \mathbb{R} \), and which satisfies \( h \cdot \mu(t) = f_1(t) \) for \( t \in \mathbb{R} \).

The idea of Lemma 1 is implicitly contained in the discussion given in §20, pp. 567–568 of [9]. For completeness we include the proof.

Proof. The \( \varepsilon \)-translation set of \( f_1 \) is

\[
T(f_1, \varepsilon) = \{ s \in \mathbb{R}: |f_1(t + s) - f_1(t)| < \varepsilon \text{ for all } t \in \mathbb{R} \}.
\]

Since \( f_1 \in \text{a.p.} \), we can find \( s_j \in T(f_1, 1/j) \) for \( j = 1, 2, \ldots, \) so that \( \lim_{j \to \infty} s_j = \infty \). Since \( g \) is bounded and satisfies (\( T \)), Lemma 3.2 of [9] guarantees that there exist a subsequence \( \{ s_{j_k} \} \) of \( \{ s_j \} \) and a bounded, uniformly continuous function \( h \) on \( \mathbb{R} \) such that

\[
\lim_{k \to \infty} \left\{ \sup_{|t| < d} |g(t + s_{j_k}) - h(t)| \right\} = 0 \quad \text{for every } d > 0.
\]

Fix \( t \in \mathbb{R} \) and observe that by (1)

\[
g \cdot \mu(t + s_{j_k}) = f_1(t + s_{j_k}) + f_2(t + s_{j_k})
\]
for \( k = 1, 2, \ldots \). Thus, if we let \( k \to \infty \) in (3) and use \( \mu \in \mathfrak{M} \), (2), \( s_h \in T(f_1, 1/j_k) \) and \( f_2(t) \to 0 \) as \( t \to \infty \), we get that \( h \ast \mu(t) = f_1(t) \). This completes the proof of Lemma 1.

The following lemma relates properties of bounded, uniformly continuous solutions of equation (1) with \( f \in \text{a.p.} \) to those of bounded, uniformly continuous solutions of the homogeneous equation

\[
g \ast \mu(t) = 0, \quad t \in \mathbb{R}.
\]

**Lemma 2.** Assume that \( \mu \in \mathfrak{M} \). If every bounded, uniformly continuous solution \( g \) of the homogeneous equation (4) is \( \text{a.p.} \), then, for each \( f \in \text{a.p.} \), every bounded, uniformly continuous solution \( g \) of equation (1) is \( \text{a.p.} \).

When \( \mu(t) = \int_{-\infty}^{t} k(s) \, ds \) with \( k \in L^1(R) \), Lemma 2 is due to Doss [4, Lemma 2]. The proof for general \( \mu \in \mathfrak{M} \) follows in exactly the same manner as is indicated by Doss in [4] for the case where \( \mu \) is absolutely continuous. Namely, use Bochner's proof of Theorem 4 in [3] with the differential operator \( \Lambda g \) replaced by the integral operator \( Kg = g \ast \mu \) throughout.

**Proof of Theorem 1.** By Lemma 1 there exists a bounded, uniformly continuous solution \( A \) of \( A \ast \mu = f_1 \) on \( \mathbb{R} \). Since \( \Lambda(\mu) \) has no finite accumulation point, every bounded, uniformly continuous solution \( h_1 \) of \( h_1 \ast \mu = 0 \) is \( \text{a.p.} \). (See, e.g., [9, Proposition 8.1].) Thus, by Lemma 2, \( h \in \text{a.p.} \). It is easy to verify that \( h \subseteq \Lambda(\mu) \cup \exp f_1 \).

Now, define \( G(t) = g(t) - h(t), t \in \mathbb{R} \). Then \( G \) satisfies \( G \ast \mu = f_2 \) on \( \mathbb{R} \), and by Theorem 3 of [7], \( G = G_1 + G_2 \), where \( G_1 \in \text{a.p.} \) with \( \exp G_1 \subseteq \Lambda(\mu) \) and \( \lim_{t \to \infty} G_2(t) = 0 \). Thus, \( g = g_1 + g_2 \) where \( g_1 \equiv G_1 - h \) and \( g_2 \equiv G_2 \) satisfy the conclusion of the theorem.

**2. Extension to distributions.** In this section we show that Theorem 1 holds if the measure \( \mu \) is replaced by a tempered distribution \( \tau \) for which, roughly, \( g \ast \tau \) makes sense for bounded \( g \).

We say that a tempered distribution \( \tau \) on \( \mathbb{R} \) satisfies property \( H \) if the convolution \( \tau \ast \phi \in L^1(\mathbb{R}) \) for every \( \phi \) in the Schwartz space \( \mathcal{S} = \mathcal{S}(\mathbb{R}) \). (The definitions of \( \mathcal{S} \), tempered distributions, the convolution \( \tau \ast \phi \), and Fourier transforms of tempered distributions are all standard; e.g., see [5].) If \( g \in L^\infty \), define the linear form \( g \ast \tau: \mathcal{S} \to \mathbb{C} \) by the formula \( g \ast \tau(\phi) = g \ast (\tau \ast \hat{\phi})(0) \) where \( \hat{\phi}(t) \equiv \phi(-t) \) and the first convolution on the right-hand side is taken in the classical sense.

**Lemma 3.** If \( \tau \) satisfies property \( H \) and \( g \in L^\infty \), then \( g \ast \tau \) is a tempered distribution.

**Proof.** Since \( |g \ast \tau(\phi)| \leq \|g\|_\infty \|\tau \ast \phi\|_1 \), the lemma is true if the mapping \( \phi \to \tau \ast \phi \) is continuous from \( \mathcal{S} \) into \( L^1 \).

Let \( \psi \) be a function in \( \mathcal{S} \) whose Fourier transform, \( \hat{\psi} \), has the following properties: \( \hat{\psi} \) is nonnegative, \( \hat{\psi}(\xi) = 1 \) if \( |\xi| < 1 \), and \( \hat{\psi}(\xi) = 0 \) if \( |\xi| > 2 \). For any positive number \( r \) define \( \psi_r \) by the formula \( \psi_r(t) = r \hat{\psi}(rt) \). Now suppose that \( \{\phi_n\} \) is a sequence in \( \mathcal{S} \), \( \phi_n \to \phi \) in \( \mathcal{S} \), and \( \tau \ast \phi_n \to f \) in \( L^1 \). Clearly \( \tau \ast \phi_n \ast \psi_r \) converges to
\( \tau \ast \phi \ast \psi_r = f \ast \psi_r \) in \( L^1 \) for each positive \( r \). Now \( \|\tau \ast \phi - f\|_1 < \|\tau \ast \phi - \\
\tau \ast \phi \ast \psi_r\|_1 + \|\tau \ast \phi \ast \psi_r - f\|_1 \), and choosing \( r \) sufficiently large it is clear that the right-hand side of the above inequality can be made arbitrarily small. Hence \( \tau \ast \phi = f \). It follows that the mapping \( \phi \rightarrow \tau \ast \phi \) has a closed graph in \( S \times L^1 \) and hence must be continuous. (For the variant of the closed graph theorem used here see, for example, [5].)

Various properties of tempered distributions satisfying property \( H \) should be clear from the definition. For example, the Fourier transform of such a distribution must be a continuous function.

Clearly, bounded measures and certain linear combinations of their derivatives are tempered distributions which satisfy property \( H \). For example, consider the distribution whose Fourier transform \( \hat{\tau} \) is given by

\[
\hat{\tau}(\xi) = \sum_{j=1}^{n} P_j(i\xi)\mu_j(\xi)
\]  

(2.1)

where the \( \mu_j \)'s are Fourier transforms of bounded measures \( \mu_j \in \mathcal{M} \), and each

\[
P_j(i\xi) = \sum_{j=0}^{\infty} a_{j,j}(i\xi)^j,
\]

is a polynomial in \( (i\xi) \). Then

\[
g \ast \tau(t) = \sum_{j=1}^{n} \int_{-\infty}^{\infty} P_j(D)g(t-s)\,d\mu_j(s)
\]

where \( P_j(D)g(t) = \sum_{l=0}^{m_j} a_{j,l}g^{(l)}(t) \) and \( g^{(l)}(t) \) is the \( l \)th derivative of \( g \).

If \( \tau \) enjoys property \( H \), the definitions of \( \Lambda(\tau) \) and property \( M\Lambda(\tau) \) are analogous to those in §1 in the case where \( \tau \in \mathcal{M} \). Namely, \( \Lambda(\tau) = \{\xi \in R: \hat{\tau}(\xi) = 0\} \), and we say that \( \tau \) satisfies property \( M\Lambda(\tau) \) if for every \( \xi \in \Lambda(\tau) \) there exists a positive integer \( m_j \) such that the function \( \hat{\tau}(\xi) = (\xi - \xi_j)^{-m_j}\hat{\tau}(\xi) \) is the Fourier transform of a tempered distribution having property \( H \) and \( \hat{\tau}(\xi_j) \neq 0 \).

**Theorem 2.** Suppose \( \tau \) satisfies property \( H \), \( g \) is a bounded solution of \( g \ast \tau = f \) and the following are true:

(i) \( \Lambda(\tau) \) has no finite accumulation point,
(ii) \( \tau \) satisfies property \( M\Lambda(\tau) \),
(iii) \( f = f_1 + f_2 \) where \( f_1 \in a.p. \) and \( f_2 \) satisfies property \( M(f_1, m) \) for \( m = \sup\{m_j\} \), where the numbers \( m_j \) are defined in the previous paragraph,
(iv) \( g \) satisfies condition \( T \).

Then \( g = g_1 + g_2 \) where \( g_1 \in a.p. \) with \( \exp g_1 \subseteq \Lambda(\tau) \cup \exp f_1 \), and \( \lim_{t \to \infty} g_2(t) = 0 \).

**Proof.** Let \( \phi(x) = e^{-x^2} \). Clearly, \( g \ast (\tau \ast \phi) = f \ast \phi + f_2 \ast \phi \). It follows from

\[
(\tau \ast \phi)'(\xi) = \hat{\tau}(\xi)\phi(\xi) \quad \text{and} \quad \Lambda(\phi) = \emptyset,
\]

that \( \Lambda(\tau \ast \phi) = \Lambda(\tau) \). Also, for each \( \epsilon > 0 \),

\[
T(f_1 \ast \phi, \epsilon) \supseteq T(f_1, \epsilon/\|\phi\|_1),
\]

so that \( f_1 \ast \phi \in a.p. \); in addition, by Lemma 4.7 of [6] \( \exp(f_1 \ast \phi) = \exp f_1 \). Finally, Lemma 3 of [7] implies that \( f_2 \ast \phi \) satisfies property \( M(f_2 \ast \phi, m) \). Thus, Theorem 2 follows from Theorem 1 with \( \mu = \tau \ast \phi \).
For an example, consider the integrodifferential equation

$$\sum_{l=0}^{n} \int_{-\infty}^{\infty} g^{(l)}(t-s) \, d\mu_l(s) = f(t), \quad t \in R,$$

(2.2)

where $\mu_l \in \mathcal{M}, \ l = 0, \ldots, n$. Clearly, this equation is a special case of $g \ast \tau = f$ where $\tau$ has property $H$ and $\hat{\tau}(\xi) = \sum_{j=0}^{n} (i\xi)^j \hat{\mu}_j(\xi)$. We say that $\xi_0 \in R$ is a zero of $\hat{\tau}(\xi)$ of multiplicity $p$ if

$$\int_{-\infty}^{\infty} |t|^p |d\mu_l(t)| < \infty \quad (l = 0, \ldots, n)$$

and if

$$(d^j/d\xi^j)\hat{\tau}(\xi) = 0 \quad (\xi = \xi_0; \ j = 0, \ldots, p - 1)$$

but

$$(d^p/d\xi^p)\hat{\tau}(\xi) \neq 0 \quad (\xi = \xi_0).$$

Assume that each zero of $\hat{\tau}(\xi)$ has finite multiplicity and let $m (\leq \infty)$ be the supremum of the multiplicities of the zeros of $\hat{\tau}(\xi)$. Then an argument similar to the proof of Lemma 5 of [7] shows that property $M\Lambda(\tau)$ holds. Thus, if $\Lambda(\tau)$ has no finite accumulation point, if $f$ is as in Theorem 2, and if $g(t)$ is a solution of (2.2) a.e. on $R$ with $g^{(l)} \in L^\infty (l = 0, \ldots, n - 1)$ and $g^{(n-1)}$ locally absolutely continuous, then $g$ has the form described in Theorem 2. We remark that the special case of equation (2.2) with $\mu_n$ the point-mass measure concentrated at $t = 0$, has been considered previously in [8, Theorem 5] (see also [9, Theorem 5a]) when $\Lambda(\tau)$ is finite.

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