A PROOF OF THE MEAN ERGODIC THEOREM FOR NONEXPANSIVE MAPPINGS IN BANACH SPACE

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Abstract. Let $C$ be a closed convex subset of a uniformly convex Banach space. Let $T: C \to C$ be a nonexpansive mapping. In this paper, we deal with the weak convergence of the arithmetical means of the sequence $T^n x$, and give a new proof of the mean ergodic theorem for nonexpansive mappings.

1. Introduction. Let $X$ be a Banach space and $C$ be a closed convex subset of $X$. A mapping $T: C \to C$ is called nonexpansive on $C$ if

$$\|Tx - Ty\| \leq \|x - y\| \quad \text{for all } x, y \in C.$$

Let $F(T)$ be the set of fixed points of $T$. If $X$ is strictly convex, $F(T)$ is closed and convex. In [1], Bâillon proved the first nonlinear ergodic theorem such that if $X$ is a real Hilbert space and $F(T) \neq \emptyset$, then for each $x \in C$, the sequence $\{S_n x\}$ defined by

$$S_n x = (1/n)(x + Tx + \cdots + T^{n-1}x)$$

converges weakly to a fixed point of $T$. It was also shown by Pazy [7] that if $X$ is a real Hilbert space and $S_n x$ converges weakly to $y \in C$, then $y \in F(T)$. These results were extended by Bâillon [2], Bruck [4] and Reich [8], [9]. In this paper, we give a new proof of the following theorem which is due to Reich [9].

Theorem. Let $X$ be a uniformly convex Banach space which has a Fréchet differentiable norm. Let $C$ be a closed convex subset of $X$ and $T: C \to C$ be a nonexpansive mapping. Then the following conditions are equivalent:

(a) $F(T) \neq \emptyset$;
(b) $\{T^n x\}$ is bounded for each $x \in C$;
(c) for each $x \in C$, $S_n T^n x$ converges weakly to $y \in C$, uniformly in $i = 1, 2, \ldots$.

2. Preliminaries. Let $X$ be a uniformly convex Banach space. The duality mapping $J$ of $X$ into $X^*$ is given by the conditions

$$(J(x), x) = \|x\|^2, \quad \|Jx\| = \|x\|.$$

If $X$ is assumed to have a Fréchet differentiable norm, $J$ is continuous. $\text{co } D$ denotes the convex hull of $D$, $\text{co } D$ the closed convex hull of $D$. For $x, y \in X$, $[x, y]$ denotes the set $\{\lambda x + (1 - \lambda)y : 0 \leq \lambda \leq 1\}$. Strong convergence is denoted by $\to$ and weak convergence is denoted by $\rightharpoonup$. 

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The following lemmas are crucial for our discussion. The next lemma is known (cf. [6]). It is a simple consequence of the definition of the modulus of convexity.

**Lemma 1.** Let $X$ be a uniformly convex Banach space with modulus of convexity $\delta$. Let $x, y \in X$. If $\|x\| < r$, $\|y\| < r$, $r < R$, and $\|x - y\| > \varepsilon > 0$, then

$$\|\lambda x + (1 - \lambda)y\| \leq r(1 - 2\lambda(1 - \lambda)\delta_R(\varepsilon))$$

for all $\lambda$: $0 < \lambda < 1$, where $\delta_R(\varepsilon) = \delta(\varepsilon/R)$.

Below, $X$ will denote a uniformly convex Banach space with modulus of convexity $\delta$.

**Lemma 2.** Let $C$ be a closed convex subset of $X$ and $T: C \to C$ be a nonexpansive mapping. Let $x \in C$, $f \in F(T)$ and $0 < \alpha < \beta < 1$. Then for each $\varepsilon > 0$, there exists $N > 0$ such that for all $n > N$,

$$\|T^k(\lambda T^n x + (1 - \lambda)f) - (\lambda T^{n+k} x + (1 - \lambda)f)\| < \varepsilon$$

for all $k > 0$ and $\lambda$: $\alpha < \lambda < \beta$.

**Proof.** Put $r = \lim_n \|T^n x - f\|$, $R = \|x - f\|$, and $c = \min\{2\lambda(1 - \lambda): \alpha < \lambda < \beta\}$. For given $\varepsilon > 0$, choose $d > 0$ so small that $r/(r + d) > 1 - c\delta_R(\varepsilon)$. Then there exists $N > 0$ such that for all $n > N$, $\|T^n x - f\| < r + d$. For $n > N$, $k > 0$ and $\alpha < \lambda < \beta$, we put $u = (1 - \lambda)(T^k z - f)$ and $v = \lambda(T^{n+k} x - T^k z)$ where $z = \lambda T^k x + (1 - \lambda) f$. Then $\|u\| \leq \lambda(1 - \lambda)\|T^k x - f\|$ and $\|v\| \leq \lambda(1 - \lambda)\|T^{n+k} x - f\|$. Suppose that $\|u - v\| = \|T^k z - (\lambda T^{n+k} x + (1 - \lambda)f)\| > \varepsilon$. Then by Lemma 1,

$$\|\lambda u + (1 - \lambda)v\| = \lambda(1 - \lambda)\|T^{n+k} x - f\|$$

$$\leq \lambda(1 - \lambda)\|T^k x - f\|/\|T^n x - f\|\!(1 - 2\lambda(1 - \lambda)\delta_R(\varepsilon))$$

$$\leq \lambda(1 - \lambda)\|T^k x - f\|/\!(1 - c\delta_R(\varepsilon)) \cdot \varepsilon.$$

Hence we have $(r + d)(1 - c\delta_R(\varepsilon)) < r < (r + d)(1 - c\delta_R(\varepsilon))$, which is a contradiction.

**Lemma 3 (Browder [3]).** Let $C$ be a closed convex subset of $X$ and $T: C \to C$ be a nonexpansive mapping. If $\{u_i\}$ is a weakly convergent sequence in $C$ with weak limit $u_0$ and if $\lim_i \|u_i - Tu_i\| = 0$, then $u_0$ is a fixed point of $T$.

3. Proof of Theorem.

**Lemma 4.** Let $C$ be a closed convex subset of $X$ and $T: C \to C$ be a nonexpansive mapping. Then for each $x \in C$ and each $n > 0$,

$$\lim_{i} \|T^k S_n T^i x - S_n T^{k+i} x\| = 0 \text{ uniformly in } k = 1, 2, \ldots \ (*)$$

**Proof.** The proof is by induction on $n$. First we prove in the case $n = 2$. Put $r = \lim_n \|T^{n+1} x - T^{n+k} x\|$, $R = \|x - T x\|$ and $x_i = T^i x$ for $i = 1, 2, \ldots$. If $r \neq 0$, then for given $\varepsilon > 0$, choose $c > 0$ so small that $r/(r + c) > 1 - \delta_R(\varepsilon)/2$. Then
there exists $N > 0$ such that for all $i > N$, $\|T^kx_i - T^kx_j\| < r + c$ for $k = 1, 2, \ldots$. If we put $u = \frac{1}{2}(T^kz - T^kx_i)$ and $v = \frac{1}{2}(T^{k+1}x_i - T^kz)$ where $i > N$, $k > 0$ and $z = \frac{1}{2}(x_i + T^kx_i)$, then

$$\|u\| < \frac{1}{2}\|z - x_i\| = \frac{1}{2}\|T^kx_i - x_i\| < \frac{1}{2}(r + c).$$

Similarly, $\|v\| < \frac{1}{2}(r + c)$. Suppose that $\|u - v\| = \|T^kz - \frac{1}{2}(T^{k+1}x_i + T^kx_i)\| > \varepsilon$. Then by Lemma 1,

$$\|\frac{1}{2}(u + v)\| = \frac{1}{2}\|T^{k+1}x_i - T^kx_i\| < \frac{1}{4}(r + c)(1 - \frac{1}{2}\delta_R(\varepsilon)),$$

which contradicts $r > (r + c)(1 - \frac{1}{2}\delta_R(\varepsilon))$. If $r = 0$, then for given $\varepsilon > 0$, choose $i > 0$ so large that $\|u\| < \varepsilon/2$ and $\|v\| < \varepsilon/2$. Hence we have $\|T^kz - \frac{1}{2}(T^{k+1}x_i + T^kx_i)\| = \|u - v\| < \|u\| + \|v\| < \varepsilon$. This completes the proof of the case $n = 2$.

Now suppose that $\lim_{i \to \infty} S_{n-1}x_i - S_{n-1}T^kx_i = 0$, uniformly in $k = 1, 2, \ldots$. We claim that $\lim_{i \to \infty} S_{n-1}T^kx_i - x_i$ exists. Put $r = \liminf_i \|S_{n-1}T^kx_i - x_i\|$. Given $\varepsilon > 0$, choose $i > 0$ such that $\|S_{n-1}T^kx_i - x_i\| < \varepsilon/2$ and $\|S_{n-1}T^kx_{i+1} - T^kS_{n-1}x_i\| < \varepsilon/2$. Then

$$\|S_{n-1}T^kx_{i+k} - x_{i+k}\| < \|S_{n-1}T^kx_{i+1} - T^kS_{n-1}x_{i+1}\| + \|T^kS_{n-1}x_{i+1} - T^kx_i\| < \varepsilon/2 + \varepsilon/2 + \varepsilon/2 = \varepsilon, \text{ for } k = 1, 2, \ldots.$$

Therefore

$$\limsup_{i \to \infty} \|S_{n-1}T^kx_i - x_i\| = \limsup_{k \to \infty} \|S_{n-1}T^kx_{i+k} - x_{i+k}\| < r + \varepsilon.$$

Since $\varepsilon$ is arbitrary, we have

$$\limsup_{i \to \infty} \|S_{n-1}T^kx_i - x_i\| < \liminf_{i \to \infty} \|S_{n-1}T^kx_i - x_i\|,$$

i.e., $\lim_i \|S_{n-1}T^kx_i - x_i\|$ exists. Now we put $r = \lim_i \|S_{n-1}T^kx_i - x_i\|$. If $r \neq 0$, for given $\varepsilon > 0$, choose $c > 0$ so small that $(r + c)/(r + 2c) > 1 - (2(n - 1)/n^2)\delta_R(\varepsilon)$. Then there exists $N > 0$ such that if $i > N$, $\|S_{n-1}T^kx_i - x_i\| < r < c$ and $\|S_{n-1}T^kx_{i+1} - T^kS_{n-1}x_{i+1}\| < c/n$. For $k > 0$ and $i > N$, we put $u = (n/(n - 1))(T^kS_{n-1}x_i - T^kx_i)$ and $v = n(S_{n-1}T^kx_{i+1} - T^kS_{n-1}x_i)$. Then

$$\|u\| < \|S_{n-1}T^kx_i - x_i\| < r + c,$$

$$\|v\| < n\|S_{n-1}T^kx_{i+1} - T^kS_{n-1}x_{i+1}\| + \|S_{n-1}T^kx_i - x_i\| < r + 2c,$$

$$\|u - v\| = (n/(n - 1))\|T^kS_{n-1}x_i - S_{n-1}T^kx_i\|.$$

Hence by the method in the proof of the case $n = 2$, we have $\|T^kS_{n-1}x_i - S_{n-1}T^kx_i\| < \varepsilon$ for $k = 1, 2, \ldots$, and all $i \geq N$. If $r = 0$, then as in the proof of the case $n = 2$ there exists $N'$ such that for each $i > N'$, $\|u\| < \varepsilon/2$ and $\|v\| < \varepsilon/2$. Therefore we have $\|T^kS_{n-1}x_i - S_{n-1}T^kx_i\| < \varepsilon$. This completes the proof.

We assume that the norm of $X$ is Fréchet differentiable. Then we have the following proposition.

**PROPOSITION 1 (cf. [4], [9]).** Let $C$ be a closed convex subset of $X$ and $T: C \to C$ be a nonexpansive mapping. If we put $W(x) = \cap_{m \geq k} \overline{co}(T^kx: k \geq m)$ for each $x \in C$, then $W(x) \cap F(T)$ is at most one point.
PROOF. Suppose that \( f, g \in W(x) \cap F(T) \) and \( f \neq g \). Put \( h = (f + g)/2 \) and \( r = \lim \|T^nx - g\| \). Since \( h \in W(x) \), \( \|h - g\| < r \). For each \( n \), we choose \( p_n \in [T^n x, h] \) such that

\[
\|p_n - g\| = \min\{\|y - g\| : y \in [T^n x, h]\}.
\]

By Theorem 2.5 of [5], \( (J(g - p_n), h - T^n x) > 0 \) where \( J \) is the duality mapping. Since \( p_n \in [T^n x, h] \), we have \( (J(g - p_n), h - T^n x) > 0 \). Suppose that

\[
\lim n \inf \|p_n - g\| = \|h - g\|.
\]

Since \( X \) is uniformly convex and \( \|p_n - g\| < \|(p_n + h)/2 - g\| < \|h - g\| \), \( p_n \) converges strongly to \( h \). Since the duality mapping \( J \) is norm-to-norm continuous, we have that for given \( \epsilon > 0 \), there exists \( N > 0 \) such that \( (J(g - h) - J(g - p_n), h - T^n x) > -\epsilon \), for all \( n > N \). Therefore we have

\[
(J(g - h), h - T^n x) = (J(g - h) - J(g - p_n), h - T^n x)
\]

\[
+ (J(g - p_n), h - T^n x)
\]

\[
> -\epsilon + 0 = -\epsilon.
\]

Then it follows that for each \( y \in \cap_m \overline{co}\{T^k x : k > m\}, (J(g - h), h - y) > 0 \). If we put \( y = g \), we have \( \|h - g\| = 0 \). This contradicts \( h \neq g \). Suppose that \( \lim \inf_n \|p_n - g\| < \|h - g\| \), then there exist \( c > 0 \) and a subsequence \( \{p_{n_k}\} \) of \( \{p_n\} \) such that \( \|p_{n_k} - g\| + c < \|h - g\| \). Put \( p_n = \alpha_n T^n x + (1 - \alpha_n)h \), for \( i = 1, 2, \ldots \). Then there exist \( \alpha > 0 \) and \( \beta < 1 \) such that \( \alpha < \alpha_i < \beta \) for all \( i \). By Lemma 2, there exists \( N > 0 \) such that if \( n > N \),

\[
\|T^k(\lambda T^n x + (1 - \lambda)h) - (\lambda T^{n+k} x + (1 - \lambda)h)\| < c
\]

for all \( \alpha < \lambda < \beta \) and for all \( k > 0 \). If we choose \( p_{n_o} \in \{p_n\} \) such that \( n_o > N \), we have

\[
\|p_{n_o+k} - g\| = \|(\alpha_{i_o} T^{n_o+k} x + (1 - \alpha_{i_o})h) - g\|
\]

\[
< \|T^{n_o+k} p_{n_o} - (\alpha_{i_o} T^{n_o+k} x + (1 - \alpha_{i_o})h)\| + \|T^k p_{n_o} - g\|
\]

\[
< c + \|p_{n_o} - g\| < \|h - g\|
\]

for \( k = 1, 2, \ldots \). Therefore we have \( p_n \neq h \) for all \( n > n_o \). It follows that \( (J(g - h), h - T^n x) < 0 \) for \( n > n_o \). Then we have \( (J(g - h), h - y) < 0 \) for all \( y \in \overline{co}(T^k x : k > n_o) \). Put \( y = f = h + (h - g) \), then \( \|h - g\| = 0 \). This contradicts \( h \neq g \).

Proof of Theorem. (a) \( \Rightarrow \) (b) is known [3]. (c) \( \Rightarrow \) (b): Suppose that for some \( x \in C \), there exists an unbounded subsequence \( \{T^n x\} \) of \( \{T^n x\} \). Since \( T \) is nonexpansive, we have that for each \( m > 0 \), the sequence \( \{S_m T^n x\} \) is also unbounded. But this contradicts the condition (c). (b) \( \Rightarrow \) (c): Since \( \{T^n x\} \) is bounded and

\[
\|TS_n T^i x - S_n T^i x\| \leq \|TS_n T^i x - S_n TT^i x\| + \|S_n TT^i x - S_n T^i x\|
\]

\[
< \|TS_n T^i x - S_n TT^i x\| + \|(1/n)\|T^{i+1+n} x - T^i x\|,
\]
there exists a sequence \( \{S_n T^k x\} \) such that \( \lim_n \|T S_n T^k x - S_n T^k x\| = 0 \). Then by Lemma 3 and Proposition 1, we have that any weakly convergent subsequence of \( \{S_n T^k x\} \) converges weakly to \( y \), i.e., \( S_n T^k x \rightharpoonup y \) where \( y = W(x) \cap F(T) \). Also by Lemma 4, we have that \( \lim_n \|T S_n T^{i+kn+ix} - S_n T^{i+kn+ix}\| = 0 \), for all \( i \) and \( k \). Therefore we have that \( S_n T^{i+kn+ix} \rightharpoonup y \) uniformly in \( k = 1, 2, \ldots \). While for \( n \) and \( m \), \( m > i_n \),

\[
S_m T^i x = \frac{1}{m} \sum_{k=0}^{m-1} T^k x_i = \frac{1}{m} \left( \sum_{k=i_n + jn}^{m-1} T^k x_i + n \left( \sum_{k=0}^{j} S_n T^{j+kn} x_j \right) + \sum_{k=0}^{i_n} T^k x_i \right)
\]

where \( m = jn + i_n + r, r < n \). Since \( \{S_n T^{i+kn} x_i\} \) converges to \( y \) uniformly with respect to \( k \), we have that \( S_m T^i x \) converges weakly to \( y \), uniformly in \( i = 1, 2, \ldots \).

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