A DIVERGENT, TWO-PARAMETER, BOUNDED MARTINGALE

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Abstract. An example is given of a divergent, uniformly bounded martingale $X = \{X_t : t \in T\}$ where the index $t$ ranges over the set $T$ of pairs of positive integers with the usual coordinatewise ordering.

This note offers an example of a divergent, uniformly bounded, two-parameter martingale which supplements the infinite-parameter example of Dieudonné [3]. Also offered here is a divergent, uniformly bounded, two-parameter, reversed martingale simpler than the one mentioned in [4].

For $m$ a positive integer and $0 < \epsilon < 1$ an $(m, \epsilon)$-daisy is a partition $\Pi$ of the universal event consisting of $m + 1$ events $C$, $B_1$, $B_2$, $B_3$, $\ldots$, $B_m$ where $C$, the center of the daisy, has probability $\epsilon$, and the $B_i$ have equal probability $(1 - \epsilon)/m$. Let $\Pi_i$ be the two-element partition consisting of $C \cup B_i$ and its complement. Plainly, the value of $P(C|\Pi_i)$ on $C \cup B_i$ is $(1 + (1 - \epsilon)/me)^{-1}$, which is now abbreviated to $c(\epsilon, m)$. Consequently, $\sup_{1 \leq i \leq m} P(C|\Pi_i) = c(\epsilon, m)$ everywhere. Indeed, a simple calculation shows that, for any pair $s$ of positive integers $a$, $b$,

$$\sup_{1 \leq i \leq m - b} P(C|\Pi_i) = c(\epsilon, m) \quad (1)$$

with probability greater than $1 - |s|/m$, where $|s|$ is $a + b$.

Let $T$ be the set of all ordered couples of positive integers endowed with the coordinatewise ordering, that is, $s < t$ if each coordinate of $t - s$ is nonnegative.

An array of partitions $\Pi_t$, $t \in T$, is based on the $(m, \epsilon)$-daisy $\Pi$ if $|t| = m$ and $t = (i, j)$ implies that $\Pi_i$ is $\Pi_j$, and if $\Pi$ is a refinement of each $\Pi_i$.

Let $\Pi^1$, $\Pi^2$, $\ldots$ form a mutually independent sequence of partitions of the universal event of a suitable probability space, such that, for each $r$, $\Pi^r$ is an $(m_r, \epsilon_r)$-daisy. Let $\{\Pi^r_t, t \in T\}$ be an array of partitions based on $\Pi^r$, and let $\mathcal{S}_r$ be the sigma-field generated by the partitions $(\Pi^r_t)$, $r = 1, 2, \ldots$ Let $A$ be the union of the centers $C'$ of the daisies $\Pi^r$.

Lemma. If $m_r \epsilon_r \to \infty$, then for each $s$

$$\sup_{1 \leq i \leq m - b} P(A|\mathcal{S}_r) = 1 \quad \text{almost surely.} \quad (2)$$

Proof. As is evident from (1), $m_r \epsilon_r \to \infty$ implies that, for each $s$, $\sup_{1 \leq i \leq m} P(C'|(\Pi^r)_i) \to 1$ in distribution as $r \to \infty$. Thus
\[
\sup_r \sup_{t>s} P(C'(\Pi'_t)) = 1 \quad \text{almost surely.}
\]

Since \( A \) includes \( C' \), \( P(A|S_r) \) exceeds \( P(C'|S_r) \), which in turn equals \( P(C'(\Pi'_t)) \) because, for each \( t \), the partitions \( (\Pi'_1), (\Pi'_2), \ldots \) are independent. Consequently, (2) must hold.

An array \( \{\Pi_t, t \in T\} \) is decreasing if \( s < t \) implies \( \Pi_s \) is a refinement of \( \Pi_t \). To obtain a decreasing array based on an \((m, \epsilon)\)-daisy \( \Pi \), first note that \( \Pi_t \) is determined for \( |t| = m \). Then \( \Pi_t \) must be the trivial partition for \( |t| > m \), and the array can be completed in various ways, for example by setting \( \Pi_t = \Pi \) for \( |t| < m \). Say the decreasing case obtains if for each \( r \) the array \( \{(\Pi'_t), t \in T\} \) introduced above is decreasing. The increasing case is defined analogously.

Plainly, \( P(A|S_r) \) is a uniformly bounded martingale or reversed martingale according as the increasing or decreasing case obtains.

**Proposition.** Suppose \( m_\epsilon \rightarrow \infty \) and \( \sum \epsilon_r < \infty \). Then, in the increasing case, \( P(A|S_r), t \in T, \) diverges with positive probability and, in the decreasing case, it diverges with probability one.

**Proof.** Since the centers \( C' \) are independent, \( \sum \epsilon_r < \infty \) implies \( 0 < PA < 1 \). Consider first the decreasing case. For any increasing sequence \( t(j) \in T, \cap S_{n(j)} \) is part of the trivial tail sigma-field of the independent sequence of partitions \( \Pi'_1, \Pi'_2, \ldots \) because \( m_\epsilon \rightarrow \infty \) and \( (\Pi')_t \) is the trivial partition for \( |t| > m_\epsilon \). Thus \( P(A|S_{n(j)}) \) converges almost surely to the constant \( PA < 1 \). This, together with (2), implies that \( P(A|S_r) \) diverges almost surely. Consider now the increasing case. If \( |t| > m_\epsilon \), then \( C'_t \), the center of \( \Pi'_t \), is \( (\Pi')_{t'} \)-measurable, and hence \( S_t \)-measurable. So if \( t(j) = (j, j) \), each \( C' \) and, hence, \( A \), is measurable relative to the limit of the \( S_{n(j)} \). Therefore, by Levy's martingale convergence theorem, \( P(A|S_{n(j)}) \) converges to zero almost everywhere off \( A \) (and to 1 almost everywhere on \( A \)). This, together with (2), implies that \( P(A|S_r) \) diverges almost everywhere on the complement of \( A \).

As is easily verified, a uniformly bounded martingale parameterised by \( T \) which diverges almost surely is \( M_t = \sum M_t^n / 2^n \), where \( (M^n_1), (M^n_2), \ldots \) is a sequence of independent copies of the martingale described above.

Of course, examples such as these indicate the necessity of some auxiliary condition to guarantee the almost sure convergence of multi-parameter martingales. The last word on this subject does not yet seem to have been said, but some such supplementary conditions can be found in the references.

**References**


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