A HUREWICZ-TYPE THEOREM FOR APPROXIMATE FIBRATIONS

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Abstract. This paper concerns conditions on point inverses which insure that a mapping between locally compact, separable, metric ANR's is an approximate fibration. Roughly a mapping is said to be \( \pi_i \)-movable [respectively, \( H_i \)-movable] provided that nearby fibers include isomorphically into mutual neighborhoods on \( \pi_i \) [resp. \( H_i \)]. An earlier result along this line is that \( \pi_i \)-movability for all \( i \) implies that a mapping is an approximate fibration. The main result here is that for a \( UV^1 \) mapping, \( \pi_i \)-movability for \( i < k - 1 \) plus \( H_k \) and \( H_{k+1} \)-movability imply \( \pi_k \)-movability of the mapping. Hence a \( UV^1 \) mapping which is \( H_k \)-movable for all \( i \) is an approximate fibration. Also, if a \( UV^1 \) mapping is \( \pi_i \)-movable for \( i < k \) and \( k \) is at least as large as the fundamental dimension of any point inverse, then it is an approximate fibration. Finally, a \( UV^1 \) mapping \( f: M^m \rightarrow N^n \) between manifolds is an approximate fibration provided that \( f \) is \( \pi_i \)-movable for all \( i < \max\{m - n, \frac{1}{2}(m - 1)\} \).

1. Introduction and statement of results. Given a proper surjective mapping \( p: E \rightarrow B \) between locally compact, separable ANR's, we are interested in conditions on the point inverses which insure that \( p \) is an approximate fibration (definition below). Earlier results in this direction involve "\( UV \)" conditions on point inverses. The mapping \( p: E \rightarrow B \) is said to be a \( k-uv \) [resp., \( k-UV \)] mapping provided that \( p \) is proper and surjective and for every \( b \) in \( B \) and every neighborhood \( U \) of \( p^{-1}(b) \), there is a neighborhood \( V \) of \( p^{-1}(b) \) in \( U \) such that the inclusion induced map \( \tilde{H}_k( V) \rightarrow \tilde{H}_k( U) \) [resp. \( \pi_k( V, e) \rightarrow \pi_k( U, e) \)] is zero [for each base point \( e \) in \( V \)]. The notation \( uv^k \) [resp., \( UV^k \)] means \( i-uv \) [resp., \( i-UV \)] for all \( i < k \). In the next section we define properties called \( \pi_i \)-movable and \( H_i \)-movable which generalize the "\( UV \)" properties by allowing nonzero images. One of the fundamental theorems on \( UV \) properties is the following Hurewicz-type theorem [L2, Theorem 4.2]. If \( p: E \rightarrow B \) is a \( UV^{k-1} \) mapping and a \( k-uv \) mapping where \( k > 2 \), then \( p \) is a \( UV^k \) mapping. The purpose of this note is to prove an analogous theorem for movable mappings.

Theorem A. Let \( p: E \rightarrow B \) be a mapping between locally compact separable ANR's. If \( p \) is \( UV^1 \), \( \pi_i \)-movable for \( i < k - 1 \), and \( H_k \) and \( H_{k+1} \)-movable where \( k > 2 \), then \( p \) is \( \pi_k \)-movable.

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As an application, we give the following improvement of [CD2, Theorem 3.7] for $UV^1$-mappings, and a result on mappings between manifolds which generalizes [L, Theorem 5.4].

**Theorem B.** If $p: E \to B$ is a $UV^1$, $\pi_i$-movable map for $i < k$ and each fiber of $p$ has fundamental dimension $\leq k$, then $f$ is an approximate fibration.

**Theorem C.** Let $f: M^m \to N^n$ be a $UV^1$ mapping between manifolds. If $f$ is $\pi_i$-movable for all $i < k - 1$ where $k > \max\{m - n + 1, (m + 1)/2\}$, then $f$ is an approximate fibration.

We use the following terminology and notation in this paper. If $p: E \to B$ is a mapping, $b \in B$ or $U \subset B$, then the fiber $p^{-1}(b)$ is denoted by $F_b$ and $p^{-1}(U)$ is denoted by $\hat{U}$. Our usual homology and cohomology groups are singular, with integral coefficient groups. If $\lambda$ is one of the usual homology, (cohomology) or homotopy functors, then $\hat{\lambda}(X)$ denotes the inverse (direct) limit of $\lambda(U)$ as $U$ ranges over the neighborhoods of $X$. An absolute neighborhood retract for metric spaces is abbreviated to ANR. A manifold is assumed to be connected and boundaryless.

2. **Movability and lifting properties.** Suppose that $p: E \to B$ is a proper surjective map. We say that $p$ has the approximate homotopy lifting property (AHLP) for a space $X$ if for each commutative diagram

$$
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{g} & E \\
\cap & \downarrow{\scriptstyle p} & \\
X \times I & \xrightarrow{\scriptstyle H} & B
\end{array}
$$

and open cover $\alpha$ of $B$, there is an extension $G: X \times I \to E$ of $g$ such that $p \circ G$ is $\alpha$-close to $H$. We say that such a $G$ is an $\alpha$-lift of $H$. If $p$ has the AHLP for all spaces $X$, $p$ is an approximate fibration.

In [CD2], it was shown that the AHLP for polyhedra can be detected by a homotopy regularity condition on fibers, which we called $k$-movability. In this paper, it will be convenient to define movability in a slightly more general setting.

Let $\Lambda_0$ be the collection of functors $\{\pi_i, H_j|_i, j = 0, 1, 2, \ldots\}$. If $\Lambda \subset \Lambda_0$, we say that $p$ is $\Lambda$-movable provided that given $b \in B$ and any neighborhood $U_0$ of $F_b$, there exist open sets $U$ and $V$ with $F_b \subset V \subset U \subset U_0$ such that for each $F_c \subset V$, each $\lambda \in \Lambda$ (and each base point in $F_c$ if relevant), the inclusion induced map sends $\hat{\lambda}(F_c)$ isomorphically onto the image of $\lambda(V)$ in $\lambda(U)$. Given such $U$, $V$, we say that $\hat{\lambda}(F_c)$ is realized as the image of $\lambda(V)$ in $\lambda(U)$.

Thus our earlier terminology, $k$-movable, is replaced in this paper by $\Lambda$-movable where $\Lambda = \{\pi_i|i < k\}$. Subject to this change the result from [CD2, Theorem 3.3] says that if $E$ and $B$ are ANR's and $f$ is $\Lambda$-movable for $\Lambda = \{\pi_i|i < k\}$, then $f$ has the AHLP for polyhedra of dimension $\leq k$.

We will need the following technical lemma.
Lemma 2.1. Let $\Lambda_1$ and $\Lambda_2$ be subsets of $\Lambda_0$. If $p : E \to B$ is $\Lambda_1$-movable and $\Lambda_2$-movable, then $p$ is $(\Lambda_1 \cup \Lambda_2)$-movable. Furthermore, if $B$ is a manifold, the open sets $U$ and $V$ may be chosen to be preimages of contractible sets.

Proof. Given $b \in B$ and $U_0 \supset F_b$, choose open sets $U_1, V_1, U_2, V_2, U_3, V_3$ with $F_b \subset V_3 \subset U_3 \subset V_2 \subset U_2 \subset V_1 \subset U_1 \subset U_0$ such that for every $F_c \subset V_3$, $\lambda \in \Lambda_1$, $\tilde{\lambda}(F_c)$ is realized as the image of $\lambda(U_3)$ in $\lambda(U_3)$ and as the image of $\lambda(V_1)$ in $\lambda(U_1)$ and such that $\mu(F_c)$ is realized as the image of $\mu(U_2)$ in $\mu(U_2)$ for every $\mu \in \Lambda_2$. It is easy to check that if $U = U_2$ and $V = V_3$, $\tilde{\lambda}(F_c)$ is realized as the image of $\lambda(V)$ in $\lambda(U)$ for each $\lambda \in \Lambda_1 \cup \Lambda_2$.

For the second conclusion, given $b \in B$ and $U_0 \supset F_b$ choose open sets $V \subset V_2 \subset U_2 \subset U \subset V_1 \subset U_1$ such that $F_b \subset V$, $U_1 \subset U_0$, $\tilde{\lambda}(F_c)$ is realized as the image of $\lambda(V_i)$ in $\lambda(U_i)$ for each $F_c \subset V_i$ ($i = 1, 2$), and $U$ and $V$ are preimages of contractible sets.

In the next section, it will be convenient to assume that $E$ and $B$ are $Q$-manifolds. A natural device is to replace $p$ by the map $p \times 1_Q : E \times Q \to B \times Q$ and appeal to Edwards’ Theorem [E]. The reader can easily provide a proof for the following lemma.

Lemma 2.2. If $p : E \to B$ is a proper map between ANR’s, then for each of the properties $P_i$ below, $p$ has $P_i$ if and only if $p \times 1_Q$ has $P_i$.

$P_1$: Being an approximate fibration.

$P_2$: Being $\lambda$-movable for some $\lambda \in \Lambda_0$.

$P_3$: Having the AHLP for a space $X$.

Lemma 2.3. Suppose that $p : E \to B$ is a map between ANR’s and that $p$ has the AHLP for polyhedra of dimension $\leq q$. If $V \subset U$ is a pair of open sets in $Y$, then $p_* : \pi_i(U, V) \to \pi_i(U, V)$ is an isomorphism for $i \leq q$ and is epic for $i = q + 1$.

The proof is a variation of a standard argument; see for example [S, Theorem 7.2.8], [L1, Corollary 2.4] and [L2, Lemma 1.2]. It uses [CD2, Lemma 1.2].

3. Proof of Theorem A. By Lemma 2.1 we may assume that both $E$ and $B$ are $Q$-manifolds. Thus, each point in $B$ has arbitrarily small contractible open neighborhoods.

Let us say that $p$ has property $i\text{-DUV}$ provided that for each $b \in B$ and each neighborhood $U_0$ of $F_b$, there are neighborhoods $V \subset U$ of $F_b$ in $U_0$ such that given any fiber $F_c$ in $V$ and any neighborhood $W_0$ of $F_c$ in $V$, there are neighborhoods $X \subset W$ of $F_c$ in $W_0$ such that the inclusion induced map $\nu : \pi_i(V, X) \to \pi_i(U, W)$ is the zero homomorphism for each base point. By Lemma 3.1 of [CD2], $p$ has property $i\text{-DUV}$ for $i < k - 1$. We wish to prove $k\text{-DUV}$ and $(k + 1)\text{-DUV}$.

Given $b \in B$ and a neighborhood $U_0$ of $F_b$, apply the hypotheses and Lemma 2.1 to choose $V \subset U$ satisfying the following properties.

(i) $\pi_{k-1}(F_c)$ is realized as the image of $\pi_{k-1}(V)$ in $\pi_{k-1}(U)$ for each $F_c \subset V$,

(ii) $\tilde{H}_F(V)$ is realized as the image of $H_i(V)$ in $H_i(U)$ for each $F_c \subset V$ and $i = k, k + 1$.
(iii) $U$ and $V$ are the preimages of contractible neighborhoods of $b$. Given a fiber $F_c \subset V$ and a neighborhood $W_0$ of $F_c$ in $V$, apply the hypotheses and the above lemmas again to choose $X \subset W$ satisfying the following properties:

(iv) $\pi_{k-1} F_c$ is realized as the image of $\pi_{k-1} X$ in $\pi_{k-1} U$,
(v) $H_i F_c$ is realized as the image $H_i X$ in $H_i W$, for $i = k, k + 1$, and
(vi) $X$ and $W$ are preimages of contractible neighborhoods of $b$.

Consider the following commutative diagram.

\[
\begin{array}{cccccc}
\pi_k(V, X) & \longrightarrow & \pi_{k-1} X & \longrightarrow & \pi_k V & \longrightarrow & \pi_{k-1} W & \longrightarrow & \pi_k(U, W) \\
& & & & & & & & \\
& & & & & & \Phi_\# & & \\
& & & & & & \Psi_\# & & \\
H_k V & \longrightarrow & H_k(V, W) & \longrightarrow & H_k(U, W) & \longrightarrow & H_k(U) \\
& & & & & & \Psi_\# \\
H_k W & \longrightarrow & H_k(U) \\
\end{array}
\]

The horizontal rows are portions of exact sequences of pairs, the vertical arrows are inclusion-induced and the diagonal arrows are Hurewicz homomorphisms. By [L2, Lemma 5.1], (iii), and (vi), $U$ and $W$ are simply connected. Also by (iii) and (vi) and Lemma 2.3, $\pi_i(U, W) = 0$ for $i < k - 1$. Thus, by [S, p. 397] $h$ is an isomorphism. By (i) and (iv) $\chi_\#| \text{im } \Phi_\#$ is monic, and by (ii) and (v) $\text{im } \Psi_\# = \text{im } \Psi_\#$. It is now an easy “diagram chasing” argument to show that $\nu_\#$ is zero and, hence, we have $k$-DUV.

Consider next the following similar diagram.

\[
\begin{array}{cccccc}
\pi_{k+1}(V, X) & \longrightarrow & H_{k+1}(V, X) & \longrightarrow & H_{k+1}(V) & \longrightarrow & H_{k+1}(U, W) \\
& & & & & & \Psi_\* \\
H_{k+1}(V) & \longrightarrow & H_{k+1}(V, W) & \longrightarrow & H_{k+1}(W) & \longrightarrow & H_{k+1}(U) \\
& & & & & & \pi_{k+1}(U, W) \\
\end{array}
\]

Since the proof of Theorem 3.3 of [CD2] really uses only $i$-DUV (rather than $i$-movability as stated), we see that $p$ has AHLP for $i^t$, $i < i < k$. As above this implies that $\pi_k(U, W) = 0$ and $h$ is an isomorphism. Also $\chi_\*| \text{im } \Phi_\*$ is monic and $\text{im } \Psi_\* = \text{im } \Psi_\* \chi_\*$ again. Hence $\nu_\#$ is zero and $(k + 1)$-DUV results.
We now apply Theorem 3.3 of [CD2] again to see that $p$ has AHLP for $i'$, $i < k + 1$. Hence $p$ is a $\pi_k$-movable map by Proposition 3.5 of [CD2], so the proof is finished.

**Corollary.** Let $p: E \to B$ be a mapping between locally compact, separable metric ANR's. If $p$ is UV$^1$ and $H_i$-movable for all $i$, then $p$ is an approximate fibration.

When Theorem A is compared to Lacher's Hurewicz-type theorem for UV properties [L2, Theorem 4.2], a discrepancy in the analogy is noticeable. There is no hypothesis in Lacher's theorem corresponding to the $H_{k+1}$-movable hypothesis in Theorem A. The reason for this difference is explained in the remark on page 51 of [CD2]. The extra hypothesis is necessary as the following example shows.

Let $f: S^3 \to S^2$ be the Hopf fibration (a generator of $\pi_3(S^2) \cong \mathbb{Z}$), and let $K$ be the complex obtained by attaching a 4-cell to the mapping cylinder $M_f$ along $S^3$ by the identity. It follows that

- $H_j(K) \cong 0$, $i < 1$,
- $H_2(K) \cong \mathbb{Z} \cong \pi_2(K)$,
- $H_3(K) \cong 0 \cong \pi_3(K)$, and
- $H_4(K) \cong \mathbb{Z}$.

Let $\alpha: S^2 \to K$ be a generator of $\pi_3(K)$ and let $M_a = (S^2 \times I \cup K)/\{(x, 1) = f(x)\}$ be the mapping cylinder of $\alpha$. Define $p: M_a \to I$ by $p((x, t)) = t$, $p(K) = 1$. Then $p$ is a $\pi_3$-movable map which is $H_3$-movable and 1-UV, but $p$ is not $\pi_3$-movable, since $\pi_3(K) = 0$, $\pi_3(S^2) \neq 0$. Thus we cannot remove the assumption that $p$ be $H_{k+1}$-movable in Theorem A.

### 4. Proofs of the applications.

**Proof of Theorem B.** Since $F_d(F_b) < k$, $\mathcal{H}^i(F_b) \cong 0$ for $i > k + 1$ and each $b \in B$. It follows from [L2, Theorem 3.1] that $p$ is an $i$-uv$(Z)$ map for $i > k + 1$. Hence $p$ is $H_i$-movable for $i > k + 1$. By Theorem A, $p$ is $\pi_i$-movable for all $i$, so $p$ is an approximate fibration by [CD2, Corollary 3.4].

**Lemma 4.1.** If $f: M^m \to N^n$ is a UV$^1$, $(\pi_i|_i < k - 1)$-movable mapping between manifolds, then for each $y \in N$, $\mathcal{H}^j(F_y) = 0$ for $j > \max\{m - k + 1, m - n + 1\}$.

**Proof.** For $n = 0, 1$, the result is contained in [LM, Theorem 1.3]. For $n > 2$, take $y \in N$ and $j > \max\{m - k + 1, m - n + 1\}$. If $U$ is a Euclidean neighborhood of $y$, then $U$ is simply connected and $\pi_{m-j}(U, U - y) = 0$. By Lemma 2.3, $\pi_{m-j}(\bar{U}, \bar{U} - F_y) = 0$. The relative Hurewicz theorem [S, p. 397] yields $H_{m-j}(\bar{U}, \bar{U} - F_y) = 0$. Since $U$ is simply connected and $f$ is UV$^1$, $\bar{U}$ is simply connected [L2, Lemma 5.1] and thus orientable [S, p. 294]. Therefore duality [S, p. 296] can be applied to give $\mathcal{H}^j(F_y) = 0$.

**Proof of Theorem C.** Let $y \in N$. By the above lemma $\mathcal{H}^j(F_y) = 0$ for all $j > \max\{m - k + 1, m - n + 1\}$. Since $k > m - n + 1$ and $k > (m + 1)/2$, $\max\{m - k + 1, m - n + 1\} < k$. Hence $\mathcal{H}^j(F_y) = 0$ for all $j > k$. By [L, Theorem 3.1], $f$ is a $j$-uv map for all $j > k$; and by Theorem A, $f$ is $(\pi_i)_{i=1}^{\infty}$-movable. Hence $f$ is an approximate fibration [CD2, Corollary 3.4].
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