A HUREWICZ-TYPE THEOREM FOR APPROXIMATE FIBRATIONS

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Abstract. This paper concerns conditions on point inverses which insure that a mapping between locally compact, separable, metric ANR's is an approximate fibration. Roughly a mapping is said to be \( \pi_i \)-movable [respectively, \( H_k \)-movable] provided that nearby fibers include isomorphically into mutual neighborhoods on \( \pi_i \) [resp. \( H_k \)]. An earlier result along this line is that \( \pi_i \)-movability for all \( i \) implies that a mapping is an approximate fibration. The main result here is that for a \( UV^1 \) mapping, \( \pi_i \)-movability for \( i < k - 1 \) plus \( H_k \) and \( H_{k+1} \)-movability imply \( \pi_k \)-movability of the mapping. Hence a \( UV^1 \) mapping which is \( H_k \)-movable for all \( i \) is an approximate fibration. Also, if a \( UV^1 \) mapping is \( \pi_i \)-movable for \( i < k \) and \( k \) is at least as large as the fundamental dimension of any point inverse, then it is an approximate fibration. Finally, a \( UV^1 \) mapping \( f: M^m \to N^n \) between manifolds is an approximate fibration provided that \( f \) is \( \pi_i \)-movable for all \( i < \max\{m - n, \frac{1}{2}(m - 1)\} \).

1. Introduction and statement of results. Given a proper surjective mapping \( p: E \to B \) between locally compact, separable ANR’s, we are interested in conditions on the point inverses which insure that \( p \) is an approximate fibration (definition below). Earlier results in this direction involve “UV” conditions on point inverses. The mapping \( p: E \to B \) is said to be a \( k-uv \) [resp., \( k-UV \)] mapping provided that \( p \) is proper and surjective and for every \( b \) in \( B \) and every neighborhood \( U \) of \( p^{-1}(b) \), there is a neighborhood \( V \) of \( p^{-1}(b) \) in \( U \) such that the inclusion induced map \( H_k(V) \to H_k(U) \) [resp. \( \pi_k(V, e) \to \pi_k(U, e) \)] is zero [for each base point \( e \) in \( V \)]. The notation \( uv^k \) [resp., \( UV^k \)] means \( i-uv \) [resp., \( i-UV \)] for all \( i < k \). In the next section we define properties called \( \pi_k \)-movable and \( H_k \)-movable which generalize the “UV” properties by allowing nonzero images. One of the fundamental theorems on UV properties is the following Hurewicz-type theorem [L2, Theorem 4.2]. If \( p: E \to B \) is a \( UV^{k-1} \) mapping and a \( k-uv \) mapping where \( k > 2 \), then \( p \) is a \( UV^k \) mapping. The purpose of this note is to prove an analogous theorem for movable mappings.

Theorem A. Let \( p: E \to B \) be a mapping between locally compact separable ANR’s. If \( p \) is \( UV^1 \), \( \pi_i \)-movable for \( i < k - 1 \), and \( H_k \) and \( H_{k+1} \)-movable where \( k > 2 \), then \( p \) is \( \pi_k \)-movable.

Received by the editors December 12, 1978 and, in revised form, March 20, 1979.

AMS (MOS) subject classifications (1970). Primary 54C10; Secondary 54C55, 55C15, 55F65, 57A15.

Key words and phrases. Approximate fibration, UV property.

1Research of both authors supported by N.S.F. contract.
As an application, we give the following improvement of [CD2, Theorem 3.7] for $UV^1$-mappings, and a result on mappings between manifolds which generalizes [L, Theorem 5.4].

**Theorem B.** If $p: E \to B$ is a $UV^1$, $\pi_{\tau}$-movable map for $i < k$ and each fiber of $p$ has fundamental dimension $< k$, then $f$ is an approximate fibration.

**Theorem C.** Let $f: M^m \to N^n$ be a $UV^1$ mapping between manifolds. If $f$ is $\pi_{\tau}$-movable for all $i < k - 1$ where $k > \max\{m - n + 1, (m + 1)/2\}$, then $f$ is an approximate fibration.

We use the following terminology and notation in this paper. If $p: E \to B$ is a mapping, $b \in B$ or $U \subset B$, then the fiber $p^{-1}(b)$ is denoted by $F_b$ and $p^{-1}(U)$ is denoted by $\tilde{U}$. Our usual homology and cohomology groups are singular, with integral coefficient groups. If $\lambda$ is one of the usual homology, (cohomology) or homotopy functors, then $\tilde{\lambda}(X)$ denotes the inverse (direct) limit of $\lambda(U)$ as $U$ ranges over the neighborhoods of $X$. An absolute neighborhood retract for metric spaces is abbreviated to ANR. A manifold is assumed to be connected and boundaryless.

2. **Movability and lifting properties.** Suppose that $p: E \to B$ is a proper surjective map. We say that $p$ has the approximate homotopy lifting property (AHLP) for a space $X$ if for each commutative diagram

$$
\begin{array}{ccc}
X \times \{0\} & \xrightarrow{g} & E \\
\cap & \downarrow{p} & \\
f \quad & \quad & B
\end{array}
$$

and open cover $\alpha$ of $B$, there is an extension $G: X \times I \to E$ of $g$ such that $p \circ G$ is $\alpha$-close to $f$. We say that such a $G$ is an $\alpha$-lift of $f$. If $p$ has the AHLP for all spaces $X$, $p$ is an approximate fibration.

In [CD2], it was shown that the AHLP for polyhedra can be detected by a homotopy regularity condition on fibers, which we called $k$-movability. In this paper, it will be convenient to define movability in a slightly more general setting.

Let $\Lambda_0$ be the collection of functors $\{\pi_i, H_j| i, j = 0, 1, 2, \ldots \}$. If $\Lambda \subset \Lambda_0$, we say that $p$ is $\Lambda$-movable provided that given $b \in B$ and any neighborhood $U_0$ of $F_b$, there exist open sets $U$ and $V$ with $F_b \subset V \subset U \subset U_0$ such that for each $F_c \subset V$, each $\lambda \in \Lambda$ (and each base point in $F_c$ if relevant), the inclusion induced map $\tilde{\lambda}(F_c)$ isomorphically onto the image of $\lambda(V)$ in $\lambda(U)$. Given such $U$, $V$, we say that $\tilde{\lambda}(F_c)$ is realized as the image of $\lambda(V)$ in $\lambda(U)$.

Thus our earlier terminology, $k$-movable, is replaced in this paper by $\Lambda$-movable where $\Lambda = \{\pi_i| i < k \}$. Subject to this change the result from [CD2, Theorem 3.3] says that if $E$ and $B$ are ANR’s and $f$ is $\Lambda$-movable for $\Lambda = \{\pi_i| i < k \}$, then $f$ has the AHLP for polyhedra of dimension $< k$.

We will need the following technical lemma.
**Lemma 2.1.** Let \( \Lambda_1 \) and \( \Lambda_2 \) be subsets of \( \Lambda_0 \). If \( p : E \to B \) is \( \Lambda_1 \)-movable and \( \Lambda_2 \)-movable, then \( p \) is \( (\Lambda_1 \cup \Lambda_2) \)-movable. Furthermore, if \( B \) is a manifold, the open sets \( U \) and \( V \) may be chosen to be preimages of contractible sets.

**Proof.** Given \( b \in B \) and \( U_0 \supseteq F_b \), choose open sets \( U_1, V_1, U_2, V_2, U_3, V_3 \) with \( F_b \subseteq V_3 \subseteq U_3 \subseteq V_2 \subseteq U_2 \subseteq V_1 \subseteq U_1 \subseteq U_0 \) such that for every \( F_c \subseteq V_3 \), \( \lambda \in \Lambda_1 \), \( \tilde{\lambda}(F_c) \) is realized as the image of \( \lambda(V_3) \) in \( \lambda(U_3) \) and as the image of \( \lambda(V_1) \) in \( \lambda(U_1) \) and such that \( \tilde{\mu}(F_c) \) is realized as the image of \( \mu(U_2) \) for every \( \mu \in \Lambda_2 \). It is easy to check that if \( U = U_2 \) and \( V = V_3 \), \( \tilde{\lambda}(F_c) \) is realized as the image of \( \lambda(V) \) in \( \lambda(U) \) for each \( \lambda \in \Lambda_1 \cup \Lambda_2 \).

For the second conclusion, given \( b \in B \) and \( U_0 \supseteq F_b \) choose open sets \( V \subseteq V_2 \subseteq U_2 \subseteq V_1 \subseteq U_1 \subseteq U_0 \) such that \( F_b \subseteq V \), \( U_1 \subseteq U_0 \), \( \tilde{\lambda}(F_c) \) is realized as the image of \( \lambda(V_1) \) in \( \lambda(U_1) \) for each \( F_c \subseteq V_i \) (\( i = 1, 2 \)), and \( U \) and \( V \) are preimages of contractible sets.

In the next section, it will be convenient to assume that \( E \) and \( B \) are \( Q \)-manifolds. A natural device is to replace \( p \) by the map \( p \times 1_Q : E \times Q \to B \times Q \) and appeal to Edwards' Theorem [E]. The reader can easily provide a proof for the following lemma.

**Lemma 2.2.** If \( p : E \to B \) is a proper map between ANR's, then for each of the properties \( P_i \) below, \( p \) has \( P_i \) if and only if \( p \times 1_Q \) has \( P_i \).

1. \( P_1 \): Being an approximate fibration.
2. \( P_2 \): Being \( \lambda \)-movable for some \( \lambda \in \Lambda_0 \).
3. \( P_3 \): Having the AHLP for a space \( X \).

**Lemma 2.3.** Suppose that \( p : E \to B \) is a map between ANR's and that \( p \) has the AHLP for polyhedra of dimension \( \leq q \). If \( V \subseteq U \) is a pair of open sets in \( Y \), then \( p_\# : \pi_i(U, V) \rightarrow \pi_i(U, V) \) is an isomorphism for \( i \leq q \) and is epic for \( i = q + 1 \).

The proof is a variation of a standard argument; see for example [S, Theorem 7.2.8], [L1, Corollary 2.4] and [L2, Lemma 1.2]. It uses [CD2, Lemma 1.2].

### 3. Proof of Theorem A

By Lemma 2.1 we may assume that both \( E \) and \( B \) are \( Q \)-manifolds. Thus, each point in \( B \) has arbitrarily small contractible open neighborhoods.

Let us say that \( p \) has property \( i \)-DUV provided that for each \( b \in B \) and each neighborhood \( U_0 \) of \( F_b \), there are neighborhoods \( V \subseteq U \) of \( F_b \) in \( U_0 \) such that given any fiber \( F_c \) in \( V \) and any neighborhood \( W_0 \) of \( F_c \) in \( V \), there are neighborhoods \( X \subseteq W \) of \( F_c \) in \( W_0 \) such that the inclusion induced map \( \nu_\# : \pi_i(X) \rightarrow \pi_i(U, W) \) is the zero homomorphism for each base point. By Lemma 3.1 of [CD2], \( p \) has property \( i \)-DUV for \( i \leq k - 1 \). We wish to prove \( k \)-DUV and \((k + 1)\)-DUV.

Given \( b \in B \) and a neighborhood \( U_0 \) of \( F_b \), apply the hypotheses and Lemma 2.1 to choose \( V \subseteq U \) satisfying the following properties.

1. \( \pi_{k-1}(F_c) \) is realized as the image of \( \pi_{k-1}(V) \) in \( \pi_{k-1}(U) \) for each \( F_c \subseteq V \),
2. \( H_iF_c \) is realized as the image of \( H_iV \) in \( H_iU \) for each \( F_c \subseteq V \) and \( i = k, k + 1 \), and
(iii) $U$ and $V$ are the preimages of contractible neighborhoods of $b$. Given a fiber $F_c \subset V$ and a neighborhood $W_0$ of $F_c$ in $V$, apply the hypotheses and the above lemmas again to choose $X \subset W$ satisfying the following properties:

(iv) $\pi_{k-1}F_c$ is realized as the image of $\pi_{k-1}X$ in $\pi_{k-1}U$,

(v) $\tilde{H}_iF_c$ is realized as the image $H_iX$ in $H_iW$, for $i = k, k + 1$, and

(vi) $X$ and $W$ are preimages of contractible neighborhoods of $b$.

Consider the following commutative diagram.

\[
\begin{array}{ccc}
\pi_k(V, X) & \rightarrow & \pi_{k-1}X \\
\downarrow & & \downarrow \Phi_\# \\
\pi_k V & \rightarrow & \pi_{k-1} W \\
\downarrow & & \downarrow \chi_\# \\
H_k V & \leftarrow & \pi_k(U, W) \\
\Psi_\# \downarrow & & h \\
H_k W & \rightarrow & H_k(U, W)
\end{array}
\]

The horizontal rows are portions of exact sequences of pairs, the vertical arrows are inclusion-induced and the diagonal arrows are Hurewicz homomorphisms. By [L2, Lemma 5.1], (iii), and (vi), $U$ and $W$ are simply connected. Also by (iii) and (vi) and Lemma 2.3, $\pi_i(U, W) = 0$ for $i < k - 1$. Thus, by [S, p. 397] $h$ is an isomorphism. By (i) and (iv) $\chi_\#|\text{im } \Phi_\#$ is monic, and by (ii) and (v) $\text{im } \Psi_\# \chi_\# = \text{im } \Psi_\#$. It is now an easy "diagram chasing" argument to show that $\nu_\#$ is zero and, hence, we have $k$-DUV.

Consider next the following similar diagram.

\[
\begin{array}{ccc}
\pi_{k+1}(V, X) & \rightarrow & H_k(X) \\
\downarrow & & \downarrow \Phi_* \\
H_{k+1}(V, X) & \rightarrow & H_{k+1}(V) \\
\downarrow & & \downarrow \chi_* \\
H_{k+1}(V) & \rightarrow & H_{k+1}(U, W) \\
\Psi_* \downarrow & & h \\
H_{k+1}(W) & \rightarrow & \pi_{k+1}(U, W)
\end{array}
\]

Since the proof of Theorem 3.3 of [CD2] really uses only $i$-DUV (rather than $i$-movability as stated), we see that $p$ has AHLP for $l^i$, $i < i < k$. As above this implies that $\pi_k(U, W) = 0$ and $h$ is an isomorphism. Also $\chi_*|\text{im } \Phi_*$ is monic and $\text{im } \Psi_*\chi_* = \text{im } \Psi_*\chi_*$ again. Hence $\nu_\#$ is zero and $(k + 1)$-DUV results.
We now apply Theorem 3.3 of [CD2] again to see that $p$ has AHLP for $i^i$, $i < k + 1$. Hence $p$ is a $\pi_k$-movable map by Proposition 3.5 of [CD2], so the proof is finished.

**Corollary.** Let $p: E \to B$ be a mapping between locally compact, separable metric ANR's. If $p$ is UV$^1$ and $H_i$-movable for all $i$, then $p$ is an approximate fibration.

When Theorem A is compared to Lacher’s Hurewicz-type theorem for UV properties [L2, Theorem 4.2], a discrepancy in the analogy is noticeable. There is no hypothesis in Lacher’s theorem corresponding to the $H_{k+1}$-movable hypothesis in Theorem A. The reason for this difference is explained in the remark on page 51 of [CD2]. The extra hypothesis is necessary as the following example shows.

Let $f: S^3 \to S^2$ be the Hopf fibration (a generator of $\pi_3(S^2) \simeq \mathbb{Z}$), and let $K$ be the complex obtained by attaching a 4-cell to the mapping cylinder $M_f$ along $S^3$ by the identity. It follows that

\[
\tilde{H}_i(K) = 0, \quad i < 1, \\
\tilde{H}_2(K) = \mathbb{Z} \simeq \pi_2(K), \\
\tilde{H}_3(K) = 0 \simeq \pi_3(K), \quad \text{and} \\
\tilde{H}_4(K) = \mathbb{Z}.
\]

Let $\alpha: S^2 \to K$ be a generator of $\pi_2(K)$ and let $M_\alpha = (S^2 \times I \cup K)/(\{(x, 1) = f(x)\})$ be the mapping cylinder of $\alpha$. Define $p: M_\alpha \to I$ by $p((x, t)) = t$, $p(K) = 1$. Then $p$ is a $\pi_2$-movable map which is $H_3$-movable and 1-UV, but $p$ is not $\pi_3$-movable, since $\pi_3(K) = 0$, $\pi_3(S^2) \neq 0$. Thus we cannot remove the assumption that $p$ be $H_{k+1}$-movable in Theorem A.

**4. Proofs of the applications.**

**Proof of Theorem B.** Since $Fd(F_b) < k$, $\tilde{H}^i(F_b) = 0$ for $i > k + 1$ and each $b \in B$. It follows from [L2, Theorem 3.1] that $p$ is an $i$-uv$(\mathbb{Z})$ map for $i > k + 1$. Hence $p$ is $H_i$-movable for $i > k + 1$. By Theorem A, $p$ is $\pi_i$-movable for all $i$, so $p$ is an approximate fibration by [CD2, Corollary 3.4].

**Lemma 4.1.** If $f: M^m \to N^n$ is a UV$^1$, $(\pi_i | i < k - 1)$-movable mapping between manifolds, then for each $y \in N$, $\tilde{H}_j(F_y) = 0$ for $j > \max(m - k + 1, m - n + 1)$.

**Proof.** For $n = 0, 1$, the result is contained in [LM, Theorem 1.3]. For $n > 2$, take $y \in N$ and $j > \max(m - k + 1, m - n + 1)$. If $U$ is a Euclidean neighborhood of $y$, then $U$ is simply connected and $\pi_{m-j}(U, U - y) = 0$. By Lemma 2.3, $\pi_{m-j}(\tilde{U}, \tilde{U} - F_y) = 0$. The relative Hurewicz theorem [S, p. 397] yields $H_{m-j}(\tilde{U}, \tilde{U} - F_y) = 0$. Since $U$ is simply connected and $f$ is UV$^1$, $\tilde{U}$ is simply connected [L2, Lemma 5.1] and thus orientable [S, p. 294]. Therefore duality [S, p. 296] can be applied to give $\tilde{H}^j(F_y) = 0$.

**Proof of Theorem C.** Let $y \in N$. By the above lemma $\tilde{H}^j(F_y) = 0$ for all $j > \max(m - k + 1, m - n + 1)$. Since $k > m - n + 1$ and $k > (m + 1)/2$, $\max(m - k + 1, m - n + 1) < k$. Hence $\tilde{H}^j(F_y) = 0$ for all $j > k$. By [L, Theorem 3.1], $f$ is a $j$-uv map for all $j > k$; and by Theorem A, $f$ is $(\pi_i | i < 1)$-movable. Hence $f$ is an approximate fibration [CD2, Corollary 3.4].
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