Reducing the codimension of Kähler immersions

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Abstract. The codimension of an immersion of a Kähler manifold may be reduced if there is a holomorphic vector field normal to the manifold.

There have been several recent results on reducing the codimension of an isometric immersion (Erbacher [2], Yau [3, p. 351]) and in particular of a minimal immersion (Colares and do Carmo [1]). In connection with these, the following result for complex geometry may be of interest.

Theorem. Let $M$ be a complex submanifold in $\mathbb{C}^n$ with normal bundle $N$ and let $V$ be an open subset of $M$. If $N|_V$ admits $r$ holomorphic sections, then $M$ lies in some $\mathbb{C}^{n-r}$.

Recall that for any complex manifold $U$ there is a splitting of the complexified tangent bundle $\mathbb{C} \otimes T(U) = T^{1,0} U \otimes T^{0,1} U$. For typographical convenience let us denote $T^{1,0} U$ by $T U$ for $U = M, V$ or $\mathbb{C}^n$ and the restriction of $T^{1,0} \mathbb{C}^n$ to a bundle over $M$ (or $V$) by $T^* M$ (or $T^* V$). In the theorem $N$ is the normal bundle of $M$ in $T^* \mathbb{C}^n$, $N = \{ \xi \in T^* M \text{ such that } \langle Z, \xi \rangle = 0 \text{ for all } Z \in TM \}$. Here we use the standard Kähler metric on $T^* \mathbb{C}^n$. $N$ is a complex bundle over $M$ but in general it is not a holomorphic bundle. Indeed, in an appropriate sense, it is an antiholomorphic subbundle of $T^* M$.

We may assume that $V$ is a coordinate patch with coordinates $z_1, \ldots, z_m$. Let $Z_k = \partial / \partial z_k$. Let $I = (i_1, \ldots, i_k)$ be a multi-index with nonnegative integer components and let $\sigma$ be a section of $T^* V$. Using the usual connection on $T \mathbb{C}^n$ we derive a new section $Z^I \sigma$ by taking the $|I|$-fold covariant derivative of $\sigma$. Here $|I| = i_1 + \cdots + i_k$ and $Z^I$ means first differentiate $i_k$ times with respect to $Z_k$, etc.

Choose some point $q \in V$. Define $S_q = \{ \xi \in T^* V_q | \xi = Z^I \sigma, I \text{ some multi-index and } \sigma \text{ some holomorphic section of } TV \}$. Then $S = \bigcup S_q$, the union taken over all points $q \in V$, is a subset of $T^* V$. We shall soon see it is a subbundle.

A section $\tau$ of $T^* V$ is said to be parallel if its covariant derivative in each direction is zero.

Lemma. Let $\tau$ be a parallel section of $T^* V$. If $\tau$ is orthogonal to $S$ at some point $q \in V$, then $\tau$ is orthogonal to $S$ at all points of $V$.

Proof. Because $\tau$ is parallel we have (1) for any local section $\sigma$ of $T^* V$, $Z^I <\sigma, \tau> = <Z^I \sigma, \tau>$. This also holds for $Z$, so (2) $<\sigma, \tau>$ is a holomorphic function whenever $\sigma$ is a local holomorphic section of $T^* V$. Now if $\sigma$ is a
holomorphic section in a neighborhood of \( q \) then, since \( \tau \) is orthogonal to \( S \) at \( q \),
\[ \langle Z^I \perp \sigma, \tau \rangle = 0 \text{ at } q \] and so by (1), \( Z^I \langle \sigma, \tau \rangle = 0 \) at \( q \) for all \( I \). By (2), \( \langle \sigma, \tau \rangle = 0 \) in this neighborhood. Now analytic continuation may be used to show that
\[ \langle Z^I \perp \sigma, \tau \rangle = 0 \] whenever \( \sigma \) is a local holomorphic section of \( TV \).

**Corollary** \( S \) and its orthogonal complement are holomorphic subbundles over \( V \) of \( TC^n \) and each is invariant under parallel translation.

**Proof.** Let \( S^\perp = \bigcup \{ \xi \in TC^n_q | \langle s, \xi \rangle = 0 \text{ for all } s \in S_q \} \), the union taken over all \( q \in V \). Any \( \xi \in TC^n_q \) has a parallel extension. Therefore the Lemma implies that \( S^\perp \) has constant fibre dimension and is invariant under parallel translation. The same must hold for \( S \). But parallel sections are holomorphic. Thus both \( S \) and \( S^\perp \) are holomorphic subbundles.

Now we have \( T'V = S \oplus S^\perp \) and this decomposition is invariant under parallel translation. It follows easily that there is a compatible orthogonal decomposition \( C^n = C^n-p \times C^p \) where \( p = \dim S^\perp \). Pick a point \( q \in V \subset C^n \). So \( q = (q_1, q_2) \) with \( q_1 \in C^n-p \) and \( q_2 \in C^p \). Since \( TV \subset S \subset TC^n-p \), it follows that \( V \subset C^n-p \times \{q_2\} \). Now if we are given, as in the Theorem, \( r \) holomorphic sections of \( T'V \) which are orthogonal to \( TV \) then \( \dim S^\perp > r \) and so \( V \) is contained in some \( C^n-r \) and the same must be true for \( M \).

The following observation relates this Theorem to the results of Erbacher and Yau. Let \( \xi \) be a holomorphic section of \( T'V \) and assume \( \xi \) is orthogonal to \( TV \). We write \( \xi = U - iJU \) where \( U \) is in the real tangent space of \( C^n \) and \( J \) gives the complex structure. Then \( V \) as a real submanifold of \( R^{2n} \) is totally geodesic in the directions \( U \) and \( JU \).

**References**


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