AN IMPROVED ESTIMATE FOR CERTAIN
DIOPHANTINE INEQUALITIES

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Abstract. Let \( \lambda_1, \ldots, \lambda_8 \) be any nonzero real numbers such that not all \( \lambda_j \) are of the same sign and not all ratios \( \lambda_j/\lambda_k \) are rational. If \( \eta, \alpha \) are any real numbers with \( 0 < \alpha < 3/70 \) then \( |\eta + \sum_{j=1}^{8} \lambda_j n^3_j| < (\max n_j)^{-\alpha} \) has infinitely many solutions in positive integers \( n_j \).

1. Introduction. Throughout \( \eta \) is any real number and \( \lambda_1, \ldots, \lambda_8 \) are any nonzero real numbers such that not all \( \lambda_j \) are of the same sign and not all ratios \( \lambda_j/\lambda_k \) are rational. Improving a result of Davenport and Heilbronn [4], Davenport and Roth [5, Theorem 2] proved:

Theorem DR. For any \( \epsilon > 0 \) the inequality \( |\eta + \sum_{j=1}^{8} \lambda_j n^3_j| < \epsilon \) has infinitely many solutions in positive integers \( n_j \).

Furthermore, Baker [1] proved that for any positive integer \( N \) the inequality

\[ |\sum_{j=1}^{8} \lambda_j p_j^3| < (\max \log p_j)^{-N} \]

has infinitely many solutions in primes \( p_j \). Results in [4] and [1] were improved and generalized by Danicic [3], Schwarz [9], Ramachandra [8], Vaughan [10], Lau and Liu [6a], [7]. In particular [7, Theorem 2] if

\[ 0 < \alpha < (\sqrt{21} - 1)/15360 \quad (1.1) \]

then the inequality \( |\eta + \sum_{j=1}^{8} \lambda_j p_j^3| < (\max p_j)^{-\alpha} \) has infinitely many solutions in primes \( p_j \). In this paper we shall prove:

Theorem. If \( 0 < \alpha < 3/70 \) then

\[ |\eta + \sum_{j=1}^{8} \lambda_j n^3_j| < (\max n_j)^{-\alpha} \quad (1.2) \]

has infinitely many solutions in positive integers \( n_j \) and no component \( n_j \) is bounded above.

Our Theorem is an improvement of Theorem DR in the error term \( \epsilon \). Also, \( \alpha < 3/70 \) is a more desirable result since it is analogous to (1.1). Furthermore the error term in (1.2) is of the right order of infinity. Indeed we may let \( \eta = 0, \lambda_1 \) be irrational and all other \( \lambda_j \) be integers then (1.2) implies that \( |\lambda_1 + (\sum_{j=2}^{8} \lambda_j n^3_j)/n^3_1| < n_1^{-3-\alpha} \) has infinitely many integer solutions \( n_1^3 \). So in view of Dirichlet's theorem [6, Theorems 193 and 194] we see that the order of infinity of the error term in (1.2) cannot be improved except the bound of \( \alpha \).
The proof of our theorem follows the basic format of the Davenport-Roth argument [5, §4]; the improvement results from a more careful treatment of the minor arcs (Lemma 9, cf. Lemma 13 of Cook [2]). An alternative method of proving (1.2) with a positive α was outlined by Vaughan [10, p. 177].

Following exactly the same argument as that of the proof of our theorem, we can improve the results in [2] by replacing the ε in Theorems 1 and 2 of [2] by $(\max_{1 \leq x \leq 6} X_j, y)^{-\beta}$ and $(\max_{1 \leq x \leq 4} X_j, y_1, y_2)^{-\beta}$ respectively, where $0 < \beta < 1/35$. We shall omit the proof of these results.

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2. Notation and definitions. Let ε be any sufficiently small positive number and x a real variable. Write $e(x) = \exp(i2\pi x)$. By n, with or without suffices, we denote positive integers. By the given hypotheses on $\lambda_j$ we may assume (cf. [2, p. 143, §2])

$$\frac{\lambda_1}{\lambda_2} < 0 \text{ and irrational.} \quad (2.1)$$

Then by Theorem 183 in [6] there are infinitely many convergents $a/q$ with $1 < q$ and

$$(a, q) = 1, \quad |\lambda_1/\lambda_2 - a/q| < 1/(2q^2). \quad (2.2)$$

Let $X$ be large so that

$$X = q^{2/3}, \quad (2.3)$$

$$I_j = I_j(x) = \int_{\nu_j X}^{2\nu_j X} e(x \lambda_j y^3) dy \quad (j = 1, 2), \quad (2.4)$$

$$S_j = S_j(x) = \begin{cases} \sum_{\nu_j X < n < 2\nu_j X} e(x \lambda_j n^3) & (j = 1, 2, 3, 4), \\ \sum_{X^{4/5} < n < 2X^{4/5}} e(x \lambda_j n^3) & (j = 5, 6, 7, 8), \end{cases} \quad (2.5)$$

where

$$\nu_1 = 1, \quad \nu_2 = |\lambda_1/\lambda_2|^{1/3}, \quad \nu_3 = |\lambda_1/ (32\lambda_3)|^{1/3}, \quad \nu_4 = |\lambda_1/ (32\lambda_4)|^{1/3}. \quad (2.6)$$

Trivially,

$$|I_j| < \nu_j X \quad (j = 1, 2), \quad |S_j| < \nu_j X \quad (j = 1, 2, 3, 4), \quad |S_j| < X^{4/5} \quad (j = 5, 6, 7, 8). \quad (2.7)$$

Put

$$V(x) = \prod_{j=1}^{8} S_j(x), \quad W(x) = I_1(x)I_2(x) \prod_{j=3}^{8} S_j(x). \quad (2.8)$$

We dissect the real line into four regions as follows.

$$\mathcal{C}_1 = \{ x : |x| < [\lambda_2]^{-1} X^{-2-\epsilon} \}, \quad \mathcal{C}_2 = \{ x : [\lambda_2]^{-1} X^{-2-\epsilon} < |x| < X^{3/70} \}, \quad (2.9)$$

$$\mathcal{C}_3 = \{ x : X^{3/70} < |x| < X \}, \quad \mathcal{C}_4 = \{ x : X < |x| \}.$$
For the given positive $a < 3/70$ let

$$M = 2\left(\max_{1 < j < 4} \nu_j\right), \quad \tau = (MX)^{-a},$$

(2.10)

$$K_u(x) = \begin{cases} 
    u^2 & \text{if } x = 0, \\
    \left(\sin(\piux) / (\pi x)^2\right)^{\frac{1}{2}} & \text{otherwise},
\end{cases}$$

(2.11)

where $u = \tau$ or $1$. Trivially,

$$K_\tau(x) < \tau^2.$$  

(2.12)

If $U > 0$, we use $V \ll U$ (or $U \gg V$) to denote $|V| < AU$, where $A$ is some positive constant which may depend on $\lambda_j, \varepsilon$ and $\eta$ only.

3. The region $\mathcal{E}_1$.

**Lemma 1.** For any real $y$,  

$$\int_{-\infty}^{\infty} e(xy)K_u(x) \, dx = \max(0, u - |y|).$$  

**Proof.** It follows from (2.11) and Lemma 4 in [4] by a simple substitution.

**Lemma 2.** For $x \in \mathcal{E}_1$, $S_j(x) = I_j(x) + O(1)$ ($j = 1, 2$).

**Proof.** This is essentially the corollary to Lemma 11 in [5].

**Lemma 3.** If $x \neq 0$ then $I_j(x) \ll X^{-2}\|x\|^{-1}$ for $j = 1, 2$.

**Proof.** By (2.4) the lemma follows from integration by parts.

**Lemma 4.**

$$\int_{\mathcal{E}_1} V(x)e(x\eta)K_\tau(x) \, dx = \int_{-\infty}^{\infty} W(x)K_\tau(x) \, dx + O(\tau^2X^{21/5-\varepsilon}).$$

**Proof.** Note that $e(x\eta) = 1 + O(|x|)$ and $S_1S_2 - I_1I_2 = S_1(S_2 - I_2) + (S_1 - I_1)I_2$. Then by (2.8), Lemma 2, (2.7) and (2.9), for $x \in \mathcal{E}_1$ we have

$$V(x)e(x\eta) - W(x) = (S_1S_2 - I_1I_2) \prod_{j=3}^{8} S_j + O(|x|) \prod_{j=1}^{8} S_j \ll X^{31/5}. \quad (3.1)$$

By (3.1), (2.12) and (2.9), we see that

$$\int_{\mathcal{E}_1} |V(x)e(x\eta) - W(x)|K_\tau(x) \, dx \ll \tau^2X^{31/5}\int_{\mathcal{E}_1} dx \ll \tau^2X^{21/5-\varepsilon}. \quad (3.2)$$

On the other hand, by Lemma 3, (2.8)2, (2.12) and (2.9),

$$\int_{x \notin \mathcal{E}_1} W(x)K_\tau(x) \, dx \ll \tau^2X^{2+16/5}\int_{x \notin \mathcal{E}_1} (X^2|x|)^{-2} \, dx \ll \tau^2X^{16/5+\varepsilon}. \quad (3.3)$$

Lemma 4 follows from (3.2) and (3.3).

**Lemma 5.** $\int_{\mathcal{E}_1} V(x)e(x\eta)K_\tau(x) \, dx \gg \tau^2X^{21/5}$. 
Proof. Let
$$\mathcal{B} = \{ n = (n_3, \ldots, n_8) : n_j X < n_j < 2n_j X \ (j = 3, 4),$$
$$X^{4/5} < n_j < 2X^{4/5} \ (j = 5, 6, 7, 8) \} \quad (3.4)$$
and \( \phi = \lambda_1 y_1 + \lambda_2 y_2 + \sum_{j=3}^{8} \lambda_j n_j^3 \) where \( y_j \) are real. It follows from (2.8)2, (2.4), (2.5) and Lemma 1 that
$$\int_{-\infty}^{\infty} W(x) K_r(x) \, dx$$
$$= \sum_{n \in \mathcal{B}} \int_{r_2 X^3}^{r_2 X^3} \int_{X^3}^{X^3} 3^{-2} (y_1 y_2)^{-2/3} \int_{-\infty}^{\infty} e(x \phi) K_r(x) \, dx \, dy_1 \, dy_2$$
$$\leq X^{-4} \sum_{n \in \mathcal{B}} \int_{r_2 X^3}^{r_2 X^3} \int_{X^3}^{X^3} \max(0, \tau - |\phi|) \, dy_1 \, dy_2. \quad (3.5)$$

If \( 3r_2^2 X^3 < y_2 < 6r_2^2 X^3, \ n \in \mathcal{B} \) and \( |\phi| < \tau/2 = o(1) \), then in view of (2.1), (3.4) and (2.6),
$$y_1 = |\lambda_2/\lambda_1| y_2 - (\lambda_3/\lambda_1) n_3^2 - (\lambda_4/\lambda_1) n_4^3 - \sum_{j=4}^{8} (\lambda_j/\lambda_1) n_j^3 + \phi/\lambda_1$$
$$< 6r_2^2 |\lambda_2/\lambda_1| X^3 + |\lambda_3/\lambda_1| 8r_2^3 X^3 + |\lambda_4/\lambda_1| 8r_2^3 X^3 + o(X^3)$$
$$= 6X^3 + X^3/4 + X^3/4 + o(X^3) < 8X^3.$$

Similarly we have \( y_1 > 3X^3 - X^3/4 - X^3/4 + o(X^3) > X^3 \). So by (3.5) and (3.4),
$$\int_{-\infty}^{\infty} W(x) K_r(x) \, dx \geq X^{-4} \sum_{n \in \mathcal{B}} \int_{3r_2 X^3}^{\tau/2} (\tau/2) \, d\phi \, dy_2 \geq \tau^2 X^{21/5}. \quad (3.6)$$

This together with Lemma 4 proves Lemma 5.

4. Some elementary lemmata. For \( j = 1, 2, 3, 4 \) and \( k = 5, 6, 7, 8 \) let
$$K(g, h) = \int_{-\infty}^{\infty} |S_j(x)|^g |S_k(x)|^h K_r(x) \, dx,$$
$$L(g, h) = \int_{-\infty}^{\infty} |S_j(x)|^g |S_k(x)|^h K_r(x) \, dx. \quad (4.1)$$

Lemma 6. \( K(2, 4) \ll X^{13/5 + \varepsilon} \) and \( K(4, 4) \ll X^{21/5 + \varepsilon} \).

Proof. These are essentially Lemmata 8 and 10 in [5] respectively.

Lemma 7. \( L(2, 4) \ll \tau X^{13/5 + \varepsilon} \) and \( L(4, 4) \ll \tau X^{21/5 + \varepsilon} \).

Proof. For the given \( j, k \) implied in \( L(2, 4) \) let
$$\mathcal{B} = \{ \xi = (n_1, \ldots, n_6) : n_j X < n_1, n_2 < 2n_j X, X^{4/5} < n_3, \ldots, n_6 < 2X^{4/5} \}$$
and \( \psi(\xi) = \lambda_1 (n_1^3 - n_2^3) + \lambda_4 (n_3^3 + n_4^3 - n_5^3 - n_6^3) \). By Lemmata 1, 6 and \( \tau < 1 \), we have
$$L(2, 4) = \sum_{\xi \in \mathcal{B}} \int_{-\infty}^{\infty} e(x \psi(\xi)) K_r(x) \, dx = \sum_{\xi \in \mathcal{B}} \max(0, \tau - |\psi(\xi)|)$$
$$< \tau \sum_{\xi \in \mathcal{B}} \max(0, 1 - |\psi(\xi)|) = \tau K(2, 4) \ll \tau X^{13/5 + \varepsilon}. \quad (4.2)$$
The inequality for $L(4, 4)$ is proved similarly.

**Lemma 8.** For $j = 1, 2$ let $\lambda_j x = \beta_j + a_j/q_j$, where $a_j, q_j$ are integers with $(a_j, q_j) = 1$. If $\beta_j \ll q_j^{-1}X^{-2-\varepsilon}$, then

(a) $S_j(x) \ll q_j^{-1/3}\min(X, X^{-2}|\beta_j|^{-1})$ when $1 < q_j < X^{1-\varepsilon}$,
(b) $S_j(x) \ll X^{3/4+\varepsilon}$ when $X^{1-\varepsilon} < q_j < X^{2+\varepsilon}$.

**Proof.** Parts (a) and (b) are essentially Lemmata 11 and 12 in [5], respectively.

**Lemma 9.** Let $\rho, \sigma$ be any constants such that $-2 - \varepsilon < \rho < \sigma$ and $0 < \sigma$. If

$$|\lambda_2|^{-1}X^\rho < |x| < X^\sigma$$  \hspace{1cm} (4.2)

then $\min(|S_1(x)|, |S_2(x)|) \ll X^{3/4+\varepsilon+\sigma/6}$.

**Proof.** This is a generalization of Lemma 13 in [5]. By Theorem 36 in [6], for each $x$ satisfying (4.2) there are integers $a_j, q_j (j = 1, 2)$ with $(a_j, q_j) = 1$ such that

$$1 < q_j < X^{2+\varepsilon}, \quad |q_j| < X^{-2-\varepsilon},$$  \hspace{1cm} (4.3)

where

$$\beta_j = \lambda_j x - a_j/q_j.$$  \hspace{1cm} (4.4)

We see that $a_2 \neq 0$. For if $a_2 = 0$ then by (4.4) and (4.3), $|\lambda_2 x| = |\beta_2| < X^{-2-\varepsilon}$. This contradicts (4.2).

If $\max(q_1, q_2) > X^{1-\varepsilon}$ then Lemma 9 follows from Lemma 8(b). Suppose that $\max(q_1, q_2) < X^{1-\varepsilon}$. Then Lemma 9 follows from Lemma 8(a) unless the bound of $S_j(x)$ in Lemma 8(a) is $> X^{3/4+\varepsilon+\sigma/6}$ for both $j = 1, 2$. If so then for both $j = 1, 2$ we have

$$q_j < X^{3/4-3\varepsilon-\sigma/2} \quad \text{and} \quad |\beta_j| < q_j^{-1/3}X^{-11/4-\varepsilon-\sigma/6}.$$  \hspace{1cm} (4.5)

By (4.4), (4.5) and (2.3),

$$|(\lambda_1/\lambda_2)a_2q_1 - a_1q_2| = q_1q_2(\lambda_1/\lambda_2)(\lambda_2 x - \beta_2) - (\lambda_1 x - \beta_1)| < q_1q_2(|\beta_1| + |\beta_2|) \ll (q_1^2/3 + q_2^2/3)\lambda x^{-11/4-\varepsilon-\sigma/6} \ll X^{-3/2-6\varepsilon-\sigma} < 1/(2q). \hspace{1cm} (4.6)$$

Now for any integers $a'$, $q'$ with $1 < q' < q$, it follows from (2.2) that

$$|q(\lambda_1/\lambda_2 - a')| > q'(\left|\frac{a'q - aq'}{qq'}\right| - \left|\frac{a}{q} - \frac{\lambda_1}{\lambda_2}\right|) > q'(\frac{1}{qq'} - \frac{1}{2q^2}) > \frac{1}{2q}. \hspace{1cm} (4.7)$$

Put $q' = |a_2q_1|$ and $a' = \pm a_1q_2$. We see that $q' \gg 1$ as $a_2 \neq 0$. So it follows from (4.6) and (4.7), that

$$|a_2q_1| > q.$$  \hspace{1cm} (4.8)

On the other hand, by (4.4), (4.5), (4.2) and (2.3),

$$|a_2q_1| = q_1q_2|\lambda_2 x - \beta_2| \ll X^{3/2-6\varepsilon-\sigma}X^\sigma < q.$$  \hspace{1cm} (4.9)

This proves Lemma 9 since (4.8) contradicts (4.9).
5. The regions $\mathcal{G}_2$, $\mathcal{G}_3$, and $\mathcal{G}_4$. Let

\[ F_1(x) = |S_1S_3S_6|^2, \quad F_2(x) = |S_2S_3S_6|^2, \quad F_3(x) = |S_3S_4S_7S_8|^2 \]  

(5.1)

and $\mathcal{R} = \sup_{x \in \mathbb{R}} \min(|S_1(x)|, |S_2(x)|)$ where $\mathbb{R}$ is some region in the real line. By (2.8) and Hölder’s inequality we have

\[ \int_{\mathcal{R}} |V(x)|K_\xi(x) \, dx < \mathcal{R} \sum_{m=1}^{2} \left( \int_{\mathcal{R}} |S_j(x)|K_\xi(x) \, dx \right)^{1/2} \left( \int_{\mathcal{R}} F_j(x)K_\xi(x) \, dx \right)^{1/2}. \]  

(5.2)

**Lemma 10.** $\int_{\mathcal{G}_2} |V(x)|K_\xi(x) \, dx \ll \tau X^{291/70 + 2\epsilon}$.

**Proof.** By (5.1), (4.1) and Hölder’s inequality we have

\[ \int_{\mathcal{G}_2} F_m(x)K_\xi(x) \, dx \ll L(2, 4) \quad (m = 1, 2) \quad \text{and} \quad \int_{\mathcal{G}_2} F_3(x)K_\xi(x) \, dx \ll L(4, 4). \]

Then by (5.2), Lemma 9 (with $\rho = -2 - \epsilon$, $\sigma = 3/70$) and Lemma 7 we have

\[ \int_{\mathcal{G}_2} |V(x)|K_\xi(x) \, dx \ll X^{3/4 + \epsilon + 1/140(\tau X^{17/5 + \epsilon})} \ll \tau X^{291/70 + 2\epsilon}. \]

This proves Lemma 10.

**Lemma 11.** Let $F(x) = \sum e^{g(x,z_1, \ldots, z_p)}$ where $f$ is any real-valued function and the summation is taken over any finite set of values $z_1, \ldots, z_p$. Then for any $B > 4/\tau$,

\[ \int_{|x| > B} |F(x)|^2K_\xi(x) \, dx \ll (\tau B)^{-1}\int_{-\infty}^{\infty} |F(x)|^2K_\xi(x) \, dx. \]

**Proof.** This is essentially Lemma 2 in [5]. See also Lemma 16 in [7].

**Lemma 12.** $\int_{\mathcal{G}_3} |V(x)|K_\xi(x) \, dx \ll X^{288/70 + 3\epsilon}$.

**Proof.** Let $\theta_0 = 3/70$ and $\theta_n = 6\epsilon + \theta_{n-1}$. Since $\theta_n \to \infty$ as $n \to \infty$ we may let $N$ be the greatest positive integer such that $\theta_N < 1$. Take $\theta_N = 1$. For each $n < N$ put $\mathcal{G}_n = \{ x : X^{\theta_{n-1}} < |x| < X^{\theta_n} \}$. By Lemma 11 (with $B = X^{\theta_{n-1}}$) and an argument similar to that in Lemma 10 we have for $m = 1, 2$

\[ \int_{\mathcal{G}_n} F_m(x)K_\xi(x) \, dx \ll \left( \tau X^{\theta_{n-1}} \right)^{-1} \int_{-\infty}^{\infty} F_m(x)K_\xi(x) \, dx \ll \left( \tau X^{\theta_{n-1}} \right)^{-1} L(2, 4) \]

as by (2.10) $X^{\theta_{n-1}} > X^{3/70} > 4/\tau$. Similarly we have

\[ \int_{\mathcal{G}_n} F_3(x)K_\xi(x) \, dx \ll \left( \tau X^{\theta_{n-1}} \right)^{-1} L(4, 4). \]

So by (5.2), Lemma 9 (with $\rho = \theta_{n-1} - \epsilon$, $\sigma = \theta_n$) and Lemma 7 we have

\[ \int_{\mathcal{G}_n} |V(x)|K_\xi(x) \, dx \ll X^{3/4 + \epsilon} \left( \tau X^{\theta_{n-1}} \right)^{-1} L(2, 4)^{1/2} L(4, 4)^{1/2} \ll X^{3/4 + 2\epsilon - 5\epsilon/6} \left( \tau X^{17/5 + \epsilon} \right)^{-1} \ll X^{83/20 + 3\epsilon - 5\epsilon/6} \ll X^{288/70 + 3\epsilon}. \]
Since $\bigcup_{n=1}^{N} S_n = \mathbb{C}$, Lemma 12 follows.

**Lemma 13.** $\int_{\mathbb{R}} |V(x)|K_\eta(x) \, dx \ll X^{16/5+\varepsilon}$.

**Proof.** By (2.8), (2.9), Hölder’s inequality, Lemma 11 (with $B = X$) and Lemma 7 we have

$$\int_{\mathbb{R}} |V(x)|K_\eta(x) \, dx \ll \left( \int_{|x| > X} |S_1S_2S_5S_6|^2 K_\eta(x) \, dx \right)^{1/2} \left( \int_{|x| > X} |S_3S_4S_7S_8|^2 K_\eta(x) \, dx \right)^{1/2} \ll (\tau X)^{-1} L(4, 4) \ll (\tau X)^{-1} X^{21/5+\varepsilon} \ll X^{16/5+\varepsilon}.$$

This proves Lemma 13.

We come now to prove our theorem. For the given $\alpha$ let $\varepsilon > 0$ satisfy $\alpha + 2\varepsilon < 3/70$. Then it follows from Lemmata 5, 10, 12 and 13 that

$$\int_{-\infty}^{\infty} V(x)e(x\eta)K_\eta(x) \, dx \gg \tau^2 X^{16/5}.$$

By Lemma 1, (2.5) and (3.4) this integral is

$$\sum_{n \in \mathbb{N}} \max_{\eta, X < n_j < 2\eta X, j = 1, 2} \left( 0, \tau - \left| \eta + \sum_{j=1}^{8} \lambda_j n_j \right| \right) \ll \tau \mathcal{R},$$

where $\mathcal{R}$ is the number of solutions $(n_1, \ldots, n_8)$ of (1.2) with $n_1, \ldots, n_8$ lying in the same range as in the last summation since by (2.10) $\tau < M^{-\alpha}(\max_{1 \leq j \leq 8} n_j / M)^{-\alpha}$. This completes the proof of our theorem.

**References**

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