THE NONEXISTENCE OF INVARIANT UNIVERSAL MEASURES ON SEMIGROUPS

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Abstract. We prove that if \( S \) is an uncountable subsemigroup of a group, then every (left or right)-translation invariant \( \sigma \)-finite measure defined on all subsets of \( S \) must be trivial. This answers a question posed by Ryll-Nardzewski and Telgarsky.

A universal measure on a set is, by definition, a (countably-additive, positive, extended real-valued) measure defined on all subsets of that set. A measure \( \mu \) on \( (X, \Sigma) \) is said to be semiregular, if whenever \( A \in \Sigma \) and \( \mu(A) > 0 \), there is \( B \in \Sigma \) such that \( B \subset A \) and such that \( 0 < \mu(B) < \infty \). It is easily seen that every \( \sigma \)-finite measure is semiregular. We start with a Proposition that will be heavily used in our Theorem. \( \aleph_1 \) denotes the first uncountable cardinal number.

**Proposition.** Every universal semiregular measure is \( \aleph_1 \)-additive.

**Proof.** Let us recall the definition of \( \aleph_1 \)-additivity. This means that whenever \( \{A_\alpha : \alpha \in J\} \) is a class of pairwise disjoint measurable sets and \( |J| = \aleph_1 \) and if \( A = \bigcup \{A_\alpha : \alpha \in J\} \) is measurable, then it is true that the measure of \( A \) is equal to the sum of the measures of \( A_\alpha \)'s. If \( \mu \) is the measure, what we demand is \( \mu(A) = \sum_{\alpha \in J} \mu(A_\alpha) \), the sum on the right being defined in the most natural way, as

\[
\sup \left\{ \sum_{\alpha \in F} \mu(A_\alpha) : F \text{ is a finite subset of } J \right\}.
\]

To prove the Proposition, let \( \mu \) be a universal semiregular measure on a set \( X \), let \( J \) be an index set with cardinality \( \aleph_1 \), let \( \{A_\alpha : \alpha \in J\} \) be a family of pairwise disjoint subsets of \( X \) indexed by \( J \) and let \( A \) be their union. We have to prove that

\[
\mu(A) = \sum_{\alpha \in J} \mu(A_\alpha). \tag{1}
\]

**Case 1.** Let \( \mu(A_\alpha) = 0 \) for every \( \alpha \in J \). Then we claim that \( \mu(A) = 0 \). If not, by the semiregularity of \( \mu \), there is some \( B \subset A \) such that \( 0 < \mu(B) < \infty \). Define a measure \( \nu \) on the index set \( J \) by the rule

\[
\nu(E) = \mu\left( \bigcup_{\alpha \in E} B \cap A_\alpha \right)
\]

It is easily checked that \( \nu \) is also countably additive. In fact it is a universal measure on \( J \) satisfying \( \nu(J) = \mu(B) \) and hence \( 0 < \nu(J) < \infty \). Further if \( \alpha \in J \) is...
any element, we have

\[ \nu(\{\alpha\}) = \mu(B \cap A_\alpha) < \mu(A_\alpha) = 0. \]

Since \( J \) is of cardinality \( n \), this contradicts a well-known theorem of Ulam (see [O, Theorem 5.6, p. 25]). This contradiction proves that \( \mu(A) \) should be zero.

**Case 2.** Let \( \mu(A_\alpha) > 0 \) for uncountably many \( \alpha \) in \( J \). Then \( \sum_{\alpha \in J} \mu(A_\alpha) \) has to be \( \infty \). Further, there is a positive integer \( n \) such that \( \mu(A_\alpha) > 1/n \) for infinitely many (in fact, uncountably many) \( \alpha \) in \( J \). Since \( A \) contains all these \( A_\alpha \)'s, the countable additivity of \( \mu \) implies that \( \mu(A) \) is also \( \infty \). Thus the equality (1) is valid in this case also.

**Case 3.** Let \( J_1 = \{\alpha \in J: \mu(A_\alpha) > 0\} \) and let \( J_1 \) be countable. Let \( B = \bigcup_{\alpha \in J_1} A_\alpha \). Then we have

\[
\begin{align*}
\mu(A) &= \mu(B) + \mu(A \setminus B) \\
&= \sum_{\alpha \in J_1} \mu(A_\alpha) + \mu(A \setminus B) \quad \text{by countable additivity} \\
&= \sum_{\alpha \in J_1} \mu(A_\alpha) + 0 \quad \text{by Case 1, since} \\
&\quad A \setminus B = \bigcup \{A_\alpha: \alpha \in J \setminus J_1\} \\
&\quad \text{and since } \mu(A_\alpha) = 0 \quad \forall \alpha \in J \setminus J_1 \\
&= \sum_{\alpha \in J} \mu(A_\alpha) \quad \text{since } \mu(A_\alpha) = 0 \quad \forall \alpha \in J \setminus J_1.
\end{align*}
\]

Thus the Proposition is proved.

**Theorem.** Let \( S \) be an uncountable semigroup embeddable in a group. Let \( \mu \) be a \( \sigma \)-finite universal right translation-invariant measure on \( S \). Then \( \mu = 0 \).

**Proof.** Let \( G \) be a group in which \( S \) is embedded as a subsemigroup. Let \( E \) be any subset of \( S \) having cardinality \( n \). Let \( H \) be the subgroup of \( G \) generated by \( E \). Let \( A \) be a subset of \( G \) meeting each left coset of \( H \) in \( G \), in exactly one point. Then one easily verifies that \( Ax \) and \( Ay \) are disjoint, whenever \( x \) and \( y \) are distinct elements of \( H \). Let

\[ A_x = (Ax) \cap S \quad (2) \]

for every \( x \) in \( H \). Then we have

\[ S = \bigcup \{A_x: x \in H\} \quad (3) \]

because we have \( G = \bigcup \{Ax: x \in H\} \). Thus (3) represents \( S \) as the union of a class of pairwise disjoint sets, indexed by the set \( H \) having cardinality \( n \). Since \( \mu \) is \( \sigma \)-finite and hence semiregular, the previous Proposition applies. Thus, we have

\[ \mu(S) = \sum_{x \in H} \mu(A_x). \quad (4) \]

Now consider two cases.

**Case 1.** Let \( \mu(A_x) = 0 \) for every \( x \) in \( H \). Then by (4) we have \( \mu(S) = 0 \) and thus the result is proved in this case.
Case 2. Let $\mu(A_x) > 0$ for some $x$ in $H$. Now if $y$ is any element of $E$, we have $A_x y = (A x \cap S)y = (A xy) \cap Sy$

\[ \subseteq Axy \cap S \] since $S$ is closed under multiplication and $y \in S$

\[ = A_{xy} \]

and therefore $\mu(A_{xy}) > \mu(A_{xy}) = \mu(A_x)$ because $\mu$ is translation-invariant, $> 0$ by our assumption in this case. Thus $\{A_{xy} : y \in E\}$ is a collection of pairwise disjoint subsets of $S$ indexed by a set of cardinality $\mathcal{N}_1$, such that every member in this collection has positive measure. This contradicts the assumption that $\mu$ is $\sigma$-finite. Hence Case 2 does not arise at all.

**Corollary.** Let $S$ be an uncountable commutative cancellative semigroup. Then every $\sigma$-finite translation-invariant universal measure on $S$ is trivial.

**Proof.** Every such semigroup can be embedded in a group and therefore our Theorem applies.

**Remarks.** The above Corollary answers a question posed in [R-T]. The special case of the above Theorem, where $S$ itself is assumed to be a group, has been proved first in [E-M] and then by a different method in [R-T].

We conclude with the following open question.

**Problem.** Is every translation-invariant universal semiregular measure on a group necessarily a multiple of the counting measure?

**References**


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