CHARACTERIZATION OF THE TRACE-CLASS

PARFENY P. SAWOROTNOW

Abstract. We characterize the trace-class \( \tau(A) \) associated with an \( H^* \)-algebra \( A \) as well as the trace-class \( (\tau c) \) of operators acting on a Hilbert space.

In this note we present a simple characterization of the trace-class \( \tau(A) \) associated with an \( H^* \)-algebra \( A \). An interesting special case of this result is a characterization of the trace-class \( (\tau c) \) [4, p. 36] of operators acting on a Hilbert space. To the best of our knowledge this is the first time a characterization of this class has been established.

An important role in the characterization is played by the property stated in the following lemma.

**Lemma 1.** Let \( A \) be a proper \( H^* \)-algebra \([1]\) and let \( \tau(A) \) be its trace-class \([5]\). Then the norm \( \tau(\cdot) \) of \( \tau(A) \) has the following property \( (x \in \tau(A)):\) \[(*) \quad \tau(x) = \operatorname{lub}\{|\operatorname{tr}(ax)|: a \in \tau(A) \text{ and } \operatorname{lub}\{|\operatorname{tr}(y*a*ay): y \in \tau(A), \tau(y*y) < 1\} < 1\} \].

**Proof.** This is a consequence of the Lemma on p. 101 of \([6]\) if we would take into account the fact that the set of the right centralizers of the form \( La: x \mapsto ax \) with \( a \in \tau(A) \) is dense in the space \( C(A) \) (defined on p. 101 of \([6]\)) and that \( ||La|| = \{\operatorname{lub} \operatorname{tr}(y*a*ay): a \in \tau(A), \tau(y*y) < 1\} \).

Our characterization is based on the notion of a trace-algebra, which we are about to define.

**Definition.** A Banach algebra \( B \) with the norm \( n(\cdot) \) is called a trace-algebra if it has an involution \( x \mapsto x^* \), a trace (a positive linear functional) \( \operatorname{tr} \) defined on it, and has the following properties (here \( x, y \) are arbitrary members of \( B \)):

1. \( \operatorname{tr}(xy) = \operatorname{tr}(yx) \).
2. \( \operatorname{tr}(x^*x) = n(x^*x) \).
3. \( n(x^*) = n(x) \).
4. \( |\operatorname{tr}(x)| < n(x) \).
5. if \( x \neq 0 \) then \( x^*x \neq 0 \).

We also make the standard assumption "\( n(xy) < n(x) \cdot n(y) \), \( x, y \in B \)," about the continuity of multiplication.

Let \( B \) be a trace-algebra. Let \( (\cdot, \cdot) \) be the scalar product on \( B \) defined in terms of the trace, \( (x, y) = \operatorname{tr}(y^*x) = \operatorname{tr}(xy^*) \) \( x, y \in B \). Then \( B \) is a pre-Hilbert space. Let \( \|\cdot\| \) be the corresponding norm and let \( A \) be the completion of \( B \) with respect to this norm.

Received by the editors April 4, 1979.

**Lemma 2.** \[ ||x|| < n(x) \] holds for each \( x \in B \).

**Proof.** Direct verification:

\[ ||x||^2 = \text{tr}(x^*x) = n(x^*) < n(x) \cdot n(x) = n(x)^2. \]

**Lemma 3.** Multiplication of \( B \) is continuous with respect to the Hilbert space norm, \( ||xy|| < ||x|| \cdot ||y|| \), for all \( x, y \in B \).

**Proof.** We verify directly:

\[ ||xy||^2 = \text{tr}(y^*x^*xy) = \text{tr}(yy^*xx) = ||x^*|| \cdot ||yy^*|| < n(x^*) \cdot n(yy^*) = n(x^*) \text{tr}(yy^*) = ||x||^2 \cdot ||y||^2. \]

**Theorem 1.** The completion \( A \) of the trace-algebra \( B \) is a proper \( H^* \)-algebra.

**Proof.** The fact that \( A \) is an \( H^* \)-algebra is easily verified. If \( x, y, z \in B \) then

\[ (xy, z) = \text{tr}(z^*xy) = (y, x^*z) = \text{tr}(yz^*x) = \text{tr}((zy^*)^*x) = (x, zy^*). \]

The involution is extendable, as an isometry, to entire \( A \); it has the same property.

Let us show that \( A \) is proper. Let \( T \) be the trivial ideal \([1, \text{p. 371}]\) of \( A \). Then \( A = T \oplus T^p \) and the orthogonal complement \( T^p \) of \( T \) is a proper \( H^* \)-algebra. If \( T \neq 0 \) then there exists some member \( a \) of \( B \) such that \( a \in T^p \). Write \( a = x + y \) with \( x \in T, y \in T^p \). Then \( x \neq 0 \) and \( ||a||^2 = ||x||^2 + ||y||^2 \). On the other hand we have \( a^*a = (x + y)^*(x + y) = y^*y \) since \( TA = AT = 0 \). This simply means that \( ||y||^2 = \text{tr}(y^*y) = \text{tr}(a^*a) = ||a||^2 \), and this is a contradiction; \( A \) is proper.

We shall refer to the algebra \( A \) above as the \( H^* \)-algebra associated with the (trace-algebra) \( B \).

**Theorem 2 (Characterization of a trace-algebra associated with an \( H^* \)-algebra).** Let \( B \) be an abstract trace-algebra whose norm \( n() \) satisfies the following condition for each \( a \in B \):

\[ n(a) = \text{lub}\{ |\text{tr}(xa)| : \text{lub}\{ \text{tr}(y^*x^*xy) < 1 \} \}. \]  

(*)

Then there exists a proper \( H^* \)-algebra \( A \) such that \( \tau(A) = B \).

**Proof.** Let \( A \) be the \( H^* \)-algebra associated with \( B \). We only need to show that \( \tau(A) = B \). Let \( x, y \in B \) and \( a = xy \). Then \( n(a) = \tau(a) \) because of Lemma 1 above.

If \( x, y \in A \sim B \) then there are sequences \( x_n, y_n \) of members of \( B \) such that \( ||x_n - x|| \to 0 \) and \( ||y - y_n|| \to 0 \). Then it is easy to check that \( \{x_ny_n\} \) is a Cauchy sequence in the norm \( n() \):

\[ n(x_ny_n - x_my_m) < n(x_n(y_n - y_m)) + n((x_n - x_m)y_m) \]

\[ = \tau(x_n(y_n - y_m)) + \tau((x_n - x_m)y_m) \]

\[ < ||x_n|| \cdot ||y_n - y_m|| + ||x_n - x_m|| \cdot ||y_m|| \to 0. \]

(Here we used Corollary 4 on p. 99 of [5]). Let \( a' \) be its limit, \( \lim_n n(a' - x_ny_n) = 0. \)

It follows that \( ||a' - x_ny_n|| \to 0. \) But \( ||xy - x_ny_n|| \to 0, \) hence \( a' = xy \), and so \( \tau(A) \subset B \).

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Conversely let \( a \in B \) and consider the functional \( f_a : S \to \text{tr}(Sa) \) on the space \( \mathcal{C}(A) \) of right centralizers of \( A \) [6, p. 101]. For each \( x \in A \) consider the centralizer \( L_x : y \to xy \) acting on \( A \). Then \( \|L_x\| = \text{lub}\{|\text{tr}(y^*x^*xy)|: y \in B, n(y^*y) < 1\} \), since \( B \) is dense in \( A \), and so \( \|f_a\| = \text{lub}\{|\text{tr}(xa)|: x \in B, \|Lx\| < 1\} = n(a) \) is finite. (The last equality follows from the condition (*) in the statement of the theorem.) Invoking Theorem 1 of [6] we conclude that \( a \in \tau(A) \). Thus \( \mathcal{T} \subset \tau(A) \).

**Corollary** (Characterization of the trace-class \((\tau c)\) of operators on a Hilbert space). For each simple trace-algebra \( B \) satisfying condition (*) of Theorem 2 above there exists a Hilbert space \( H \) such that \( B \) is isomorphic and isometric to the trace-class \((\tau c)\) [4, p. 36] of operators acting on \( H \).

**Proof.** It is easy to see that the algebra \( A \) associated with \( B \) is simple. It follows then from the second structure theorem for \( H^*\)-algebras (Theorem 4.3 on p. 380 of [1]) that \( A \) can be identified with the algebra \((ac)\) [4, p. 29] of Hilbert-Schmidt operators acting on the Hilbert space \( H = L^2(\Gamma) \), where \( \Gamma = \{e_a\} \) is a maximal family of primitive doubly orthogonal selfadjoint idempotents of \( A \). Then \( B \) could be identified with the trace-class \((\tau c)\) of operators acting on \( H \).

**References**


**Department of Mathematics, The Catholic University of America, Washington, D.C. 20064**