CHARACTERIZATION OF THE TRACE-CLASS

PARFENY P. SAWOROTNOW

ABSTRACT. We characterize the trace-class \( \tau(A) \) associated with an \( H^* \)-algebra \( A \) as well as the trace-class \( (\tau_c) \) of operators acting on a Hilbert space.

In this note we present a simple characterization of the trace-class \( \tau(A) \) associated with an \( H^* \)-algebra \( A \). An interesting special case of this result is a characterization of the trace-class \( (\tau_c) \) [4, p. 36] of operators acting on a Hilbert space. To the best of our knowledge this is the first time a characterization of this class has been established.

An important role in the characterization is played by the property stated in the following lemma.

**Lemma 1.** Let \( A \) be a proper \( H^* \)-algebra [1] and let \( \tau(A) \) be its trace-class [5]. Then the norm \( \tau(\ ) \) of \( \tau(A) \) has the following property \( (x \in \tau(A)) \): \( (*) \) \( \tau(x) = \text{lub}\{\text{tr}(ax) : a \in \tau(A) \text{ and } \text{lub}\{\text{tr}(y^*a^*ay) : y \in \tau(A), \tau(y^*y) < 1\} < 1}\} \).

**Proof.** This is a consequence of the Lemma on p. 101 of [6] if we would take into account the fact that the set of the right centralizers of the form \( La : x \rightarrow ax \) with \( a \in \tau(A) \) is dense in the space \( C(A) \) (defined on p. 101 of [6]) and that \( \|La\| = \{\text{lub} \text{tr}(y^*a^*ay) : a \in \tau(A), \tau(y^*y) < 1\} \).

Our characterization is based on the notion of a trace-algebra, which we are about to define.

**Definition.** A Banach algebra \( B \) with the norm \( n(\ ) \) is called a trace-algebra if it has an involution \( x \rightarrow x^* \), a trace (a positive linear functional) \( \text{tr} \) defined on it, and has the following properties (here \( x, y \) are arbitrary members of \( B \)):

1. \( \text{tr}(xy) = \text{tr}(yx) \).
2. \( \text{tr}(x^*x) = n(x^*x) \).
3. \( n(x^*) = n(x) \).
4. \( |\text{tr}(x)| < n(x) \).
5. if \( x \neq 0 \) then \( x^*x \neq 0 \).

We also make the standard assumption "\( n(xy) < n(x) \cdot n(y), x, y \in B \)," about the continuity of multiplication.

Let \( B \) be a trace-algebra. Let \( (\ , \ ) \) be the scalar product on \( B \) defined in terms of the trace, \( (x, y) = \text{tr}(y^*x) = \text{tr}(xy^*) (x, y \in B) \). Then \( B \) is a pre-Hilbert space. Let \( \| \| \) be the corresponding norm and let \( A \) be the completion of \( B \) with respect to this norm.

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Lemma 2. "\(\|x\| < n(x)\)" holds for each \(x \in B\).

Proof. Direct verification:
\[
\|x\|^2 = \text{tr}(x^*x) = n(x^*x) < n(x)n(x) = n(x)^2.
\]

Lemma 3. Multiplication of \(B\) is continuous with respect to the Hilbert space norm, \(\|xy\| < \|x\| \cdot \|y\|\), for all \(x, y \in B\).

Proof. We verify directly:
\[
\|xy\|^2 = \text{tr}(y^*x^*xy) = \text{tr}(yy^*x^*x) = \|x\|^2 \text{tr}(yy^*) = \|x\|^2 \cdot \|y\|^2.
\]

Theorem 1. The completion \(A\) of the trace-algebra \(B\) is a proper \(H^*\)-algebra.

Proof. The fact that \(A\) is an \(H^*\)-algebra is easily verified. If \(x, y, z \in B\) then
\[
(xy, z) = \text{tr}(z^*xy) = (y, x^*z) = \text{tr}(yz^*x) = \text{tr}(z^*y^*x) = (x, zy^*).
\]
The involution is extendable, as an isometry, to entire \(A\); it has the same property.

Let us show that \(A\) is proper. Let \(T\) be the trivial ideal \([1, \text{p. 371}]\) of \(A\). Then \(A = T \oplus T^p\) and the orthogonal complement \(T^p\) of \(T\) is a proper \(H^*\)-algebra. If \(T \neq 0\) then there exists some member \(a\) of \(B\) such that \(a \not\in T^p\). Write \(a = x + y\) with \(x \in T, y \in T^p\). Then \(x \neq 0\) and \(\|a\|^2 = \|x\|^2 + \|y\|^2\). On the other hand we have \(a^*a = (x + y)^*(x + y) = y^*y\) since \(TA = AT = 0\). This simply means that \(\|y\|^2 = \text{tr}(y^*y) = \text{tr}(a^*a) = \|a\|^2\), and this is a contradiction; \(A\) is proper.

We shall refer to the algebra \(A\) above as the \(H^*\)-algebra associated with the (trace-algebra) \(B\).

Theorem 2 (Characterization of a trace-algebra associated with an \(H^*\)-algebra). Let \(B\) be an abstract trace-algebra whose norm \(n()\) satisfies the following condition for each \(a \in B\):
\[
n(a) = \text{lub}\{\text{tr}(xa)\} = \text{lub}\{\text{tr}(y^*x^*xy) < 1\} \quad (\star)\]

Then there exists a proper \(H^*\)-algebra \(A\) such that \(\tau(A) = B\).

Proof. Let \(A\) be the \(H^*\)-algebra associated with \(B\). We only need to show that \(\tau(A) = B\). Let \(x, y \in B\) and \(a = xy\). Then \(n(a) = \tau(a)\) because of Lemma 1 above. If \(x, y \in A \sim B\) then there are sequences \(x_n, y_n\) of members of \(B\) such that \(\|x_n - x\| \rightarrow 0\) and \(\|y - y_n\| \rightarrow 0\). Then it is easy to check that \(\{x_ny_n\}\) is a Cauchy sequence in the norm \(n()\):
\[
n(x_ny_n - x_my_m) < n(x_n(y_n - y_m)) + n((x_n - x_m)y_m) = \tau(x_n(y_n - y_m)) + \tau((x_n - x_m)y_m) \leq \|x_n\| \cdot \|y_n - y_m\| + \|x_n - x_m\| \cdot \|y_m\| \rightarrow 0.
\]
(Here we used Corollary 4 on p. 99 of \([5]\).) Let \(a'\) be its limit, \(\lim_n n(a' - x_ny_n) = 0\). It follows that \(\|a' - x_ny_n\| \rightarrow 0\). But \(\|xy - x_ny_n\| \rightarrow 0\), hence \(a' = xy\), and so \(\tau(A) \subset B\).
Conversely let \( a \in B \) and consider the functional \( f_a : S \to \text{tr}(Sa) \) on the space \( \mathcal{C}(A) \) of right centralizers of \( A \) [6, p. 101]. For each \( x \in A \) consider the centralizer \( L_x : y \to xy \) acting on \( A \). Then \( ||L_x|| = \text{lub}(|\text{tr}(y^*x^*xy)| : y \in B, \ n(y^*y) < 1} \), since \( B \) is dense in \( A \), and so \( ||f_a|| = \text{lub}(|\text{tr}(xa)| : x \in B, \ ||x|| < 1} = n(a) \) is finite. (The last equality follows from the condition (\( \ast \)) in the statement of the theorem.) Invoking Theorem 1 of [6] we conclude that \( a \in \tau(A) \). Thus \( B \subset \tau(A) \).

**Corollary** (Characterization of the trace-class (\( \tau_c \)) of operators on a Hilbert space). For each simple trace-algebra \( B \) satisfying condition (\( \ast \)) of Theorem 2 above there exists a Hilbert space \( H \) such that \( B \) is isomorphic and isometric to the trace-class (\( \tau_c \)) [4, p. 36] of operators acting on \( H \).

**Proof.** It is easy to see that the algebra \( A \) associated with \( B \) is simple. It follows then from the second structure theorem for \( H^*\)-algebras (Theorem 4.3 on p. 380 of [1]) that \( A \) can be identified with the algebra (\( \sigma_c \)) [4, p. 29] of Hilbert-Schmidt operators acting on the Hilbert space \( H = L^2(\Gamma) \), where \( \Gamma = \{e_a\} \) is a maximal family of primitive doubly orthogonal selfadjoint idempotents of \( A \). Then \( B \) could be identified with the trace-class (\( \tau_c \)) of operators acting on \( H \).

**References**


**Department of Mathematics, The Catholic University of America, Washington, D. C. 20064**