

INTERPOLATION FAILS FOR
 THE SOUSLIN-KLEENE CLOSURE OF
 THE OPEN SET QUANTIFIER LOGIC

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ABSTRACT. In this paper we show that the Souslin-Kleene closure of the open set quantifier logic fails to have interpolation. We also show that the notion of a T_0 -topological space is not definable in this logic. This gives a more natural proof that it is strictly weaker than the interior operator logic.

The questions of whether the Souslin-Kleene closure of the open set quantifier logic is the interior operator logic or even has interpolation come naturally from the work of the author in [3], [4], and [5]. We begin by giving the formal definitions and then the three results which settle these questions.

DEFINITION. The *Souslin-Kleene closure*, $\Delta(\mathcal{L}^*)$, of a logic \mathcal{L}^* is the logic formed by adding the complementary pseudo-elementary classes to the elementary classes.

That is, Ω and Ω^c are $\text{PC}_{\mathcal{L}^*}(L)$ -classes if and only if they are $\text{EC}_{\Delta(\mathcal{L}^*)}(L)$ -classes. See [1] for further background.

DEFINITION. Take a structure \mathfrak{A} and $q \subseteq \mathcal{P}(A)$ and form (\mathfrak{A}, q) . If q is a topology on A then (\mathfrak{A}, q) is called *topological*.

DEFINITION. The *open set quantifier logics* $\mathcal{L}(Q)$ and $\mathcal{L}(Q^n)_{n \in \omega}$ are formed by adding quantifiers Qx and $Q\vec{x}$, $n \in \omega$, to first order logic where the interpretations of $Qx\varphi(x)$ and $Q\vec{x}\varphi(\vec{x})$, respectively, are that the sets defined by $\varphi(x)$ and $\varphi(\vec{x})$ are open in the topology and the n th product topology. For further background see [3] and [4].

We are now ready to state and prove the main theorems of this paper.

THEOREM 1. For each (\mathfrak{A}, q) where \mathfrak{A} is an L -structure there is an $L^\# \supseteq L$ and an extension $\mathfrak{A}^\#$ of \mathfrak{A} to $L^\#$ such that if $(\mathfrak{B}, r) \equiv_{\mathcal{L}(Q)} (\mathfrak{A}^\#, q)$ then

$$(\mathfrak{B} \upharpoonright L, r) \equiv_{\Delta(\mathcal{L}(Q))} (\mathfrak{A}, q).$$

PROOF. This result is a straightforward application of the definition of $\Delta(\mathcal{L}(Q))$.

COUNTEREXAMPLE 2. The counterexample to interpolation for $\Delta(\mathcal{L}(Q^n_{n \in \omega}))$ is the same as the one for $\mathcal{L}(Q)$ as presented in [3] and [4].

We will assume that interpolation holds and derive a contradiction. Let $L_1 = \{B(x), C(x), R(x)\}$ and $L_2 = \{B(x), C(x), P(x)\}$. We define $\varphi(R)$ to be

$$\neg QxB(x) \wedge \forall y(B(y) \leftrightarrow C(y) \vee R(y)) \wedge QxR(x)$$

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and $\psi(P)$ to be

$$\forall x(C(x) \rightarrow P(x)) \rightarrow \neg Qx(P(x) \wedge B(x)).$$

One easily sees that $\vDash \varphi(R) \rightarrow \psi(P)$. Take $A = \mathbb{N}$, i.e. the set of natural numbers

$$B^{\mathfrak{A}} = \{2n | n \in \mathbb{N}\}, \quad C^{\mathfrak{A}} = \{4n | n \in \mathbb{N}\}.$$

Define (\mathfrak{A}, q) to be $\langle A, B^{\mathfrak{A}}, C^{\mathfrak{A}}, \{\emptyset, \mathbb{N}\} \rangle$.

Now since we have assumed that $\Delta(\mathcal{L}(Q))$ has interpolation there is a $\theta \in \Delta(\mathcal{L}(Q))$ such that $\vDash \varphi(R) \rightarrow \theta$ and $\vDash \neg \psi(P) \rightarrow \neg \theta$.

Without loss of generality we assume that (\mathfrak{A}, q) models θ since the argument in the alternate case is entirely analogous.

Now expand \mathfrak{A} to an $L^{\#}$ -structure $\mathfrak{A}^{\#}$ as in Theorem 1. We then will expand q to a $q^{\#}$ and define a $P^{\mathfrak{A}^{\#}}$ such that $(\mathfrak{A}^{\#}, q) <_{\mathcal{L}(Q)} (\mathfrak{A}^{\#}, q^{\#})$ and $(\mathfrak{A}, P^{\mathfrak{A}^{\#}}, q^{\#}) \vDash \neg \psi(P)$. This implies that $(\mathfrak{A}, q^{\#}) \equiv_{\Delta(\mathcal{L}(Q))} (\mathfrak{A}, q)$ and $(\mathfrak{A}, q^{\#}) \vDash \neg \theta$ which is a contradiction.

Let $\psi_i(x)$, $i \in \omega$, enumerate the $L_A^{\#}(Q)$ definable nonopen sets of $(\mathfrak{A}^{\#}, q)$. We proceed by induction. For 0 we pick an x and y such that

$$y \in [\psi_0(x)]^{(\mathfrak{A}^{\#}, q)}$$

and

$$x \in B^{\mathfrak{A}^{\#}} - [\psi_0(x)]^{(\mathfrak{A}^{\#}, q)}$$

if possible, otherwise

$$x \in A - [\psi_0(x)]^{(\mathfrak{A}^{\#}, q)}.$$

Assume we have picked the sequences y_0, \dots, y_n and x_0, \dots, x_n . We will now choose x_{n+1} and y_{n+1} as follows. Choose, if possible, $x \in B^{\mathfrak{A}^{\#}} - [\psi_{n+1}(x)]^{(\mathfrak{A}^{\#}, q)}$ such that $x \neq y_i$ for $0 \leq i \leq n$. Otherwise pick y such that

$$y \in [\psi_{n+1}(x)]^{(\mathfrak{A}^{\#}, q)} \cap B^{\mathfrak{A}^{\#}}$$

and

$$y \neq x_i \quad \text{for } 0 \leq i \leq n.$$

This is possible since otherwise $[\psi_{n+1}(x)]^{(\mathfrak{A}^{\#}, q)} \cap B^{\mathfrak{A}^{\#}}$ and $B^{\mathfrak{A}^{\#}} - [\psi_{n+1}(x)]^{(\mathfrak{A}^{\#}, q)}$ would be finite which would imply that $B^{\mathfrak{A}^{\#}}$ is finite which is false.

Let $P^{\mathfrak{A}^{\#}}$ be $(C^{\mathfrak{A}^{\#}} \cup \{x_i\}_{i \in \omega}) \cap B^{\mathfrak{A}^{\#}}$ and let $q^{\#}$ be the topology generated by $q \cup \{P^{\mathfrak{A}^{\#}}\}$. We claim that $(\mathfrak{A}^{\#}, q) <_{\mathcal{L}(Q)} (\mathfrak{A}^{\#}, q^{\#})$ and that $(\mathfrak{A}^{\#}, P^{\mathfrak{A}^{\#}}, q^{\#}) \vDash \neg \psi(P)$. The second clause is straightforward. We prove the first by induction on the number of occurrences of Qx .

If $(\mathfrak{A}^{\#}, q) \vDash Qx\varphi(x)$ then $(\mathfrak{A}^{\#}, q^{\#}) \vDash Qx\varphi(x)$ since $q \subseteq q^{\#}$, thus assume $(\mathfrak{A}^{\#}, q^{\#}) \vDash Qx\varphi(x)$ and $(\mathfrak{A}^{\#}, q) \vDash \neg Qx\varphi(x)$ and derive a contradiction. Thus $[\varphi(x)]^{(\mathfrak{A}^{\#}, q)} = P^{\mathfrak{A}^{\#}}$. But there is a k such that $[\varphi(x)]^{(\mathfrak{A}^{\#}, q)} = [\psi_k(x)]^{(\mathfrak{A}^{\#}, q)}$ so by the definition of $P^{\mathfrak{A}^{\#}}$ either

$$x_k \in P^{\mathfrak{A}^{\#}} - [\psi_k(x)]^{(\mathfrak{A}^{\#}, q)}$$

or

$$y_k \in [\psi_k(x)]^{(\mathfrak{A}^*, q)} - P^{\mathfrak{A}^*}.$$

Hence a contradiction.

REMARK. The analogous result for $\Delta(\mathcal{L}(Q^n)_{n \in \omega})$ can be proved by the same method. Also this result shows that the interior operator logics $\mathcal{L}(I)$ and $\mathcal{L}(I^n)_{n \in \omega}$ as defined in [3] and [5] strictly contain $\Delta(\mathcal{L}(Q))$ and $\Delta(\mathcal{L}(Q^n)_{n \in \omega})$, respectively, since they both have interpolation by [5].

By [2] we know that because $\Delta(\mathcal{L}(Q))$ and $\Delta(\mathcal{L}(Q^n)_{n \in \omega})$ do not have interpolation they do not have a Beth definability theorem.

However this result of strict containment can be improved by giving a more natural counterexample in the topological sense.

DEFINITION. A topological space is called T_0 (Minkowski), if and only if for each $x \neq y$ there is an open set containing one but not the other.

We can equivalently define a T_0 -space as a space where unequal points have unequal closures. See [6].

The class of T_0 -spaces is the class of models of the $\mathcal{L}(I)$ sentence

$$\forall x \forall y (x \neq y \rightarrow (Iy(y \neq x) \vee Ix(x \neq y))).$$

However we will now prove that the class of T_0 models is not a basic elementary class of $\Delta(\mathcal{L}(Q))$.

Take \mathfrak{A} to be ${}^2\mathbf{N} = \{f \mid f: \{0, 1\} \rightarrow \mathbf{N}\}$ and $L = \emptyset$.

Define a pseudometric by $d(x, y) = |x(0) - y(0)|$. Then the topology that d generates, call it q , is generated by the closures of points and every open set is infinite. (\mathfrak{A}, q) also is not a T_0 -space since the closure of a point, which is infinite, is the closure of any point in it.

Now we will construct the counterexample using the following theorem.

THEOREM 3. *There is a topology $q^\#$ such that $(\mathfrak{A}, q^\#)$ is a T_0 -topology and $(\mathfrak{A}, q) \equiv_{\Delta(\mathcal{L}(Q))} (\mathfrak{A}, q^\#)$.*

PROOF. To show this result we expand \mathfrak{A} and L to $\mathfrak{A}^\#$ and $L^\#$ as in Theorem 1 (taking pains to add functions to the language to pick out noninterior points from definable nonopen sets as in [3]).

Given a pair a, b we will define a topology $q_{\langle a, b \rangle}$ such that $(\mathfrak{A}^\#, q) <_{\mathcal{L}(Q)} (\mathfrak{A}^\#, q_{\langle a, b \rangle})$, a and b have unequal closures, and $q_{\langle a, b \rangle}$ is generated by the closures of points and every open set is infinite. This is the same topological property of (\mathfrak{A}, q) which we use.

We then iterate this construction through all distinct pairs and take the union (see [3]) which will be T_0 and satisfy the conclusion to the theorem.

Define $x_{-1} = a$ and $y_{-1} = b$. Take h to be a bijection from \mathbf{N} into $\mathbf{N} \times \mathbf{N} \times 2$. Let $\psi_i(x)$, $i \in \omega$, enumerate the $L_A^\#(Q)$ definable nonopen sets and let θ_i , $i \in \omega$, enumerate the closures of points, which is a basis for q .

Assume we have defined $x_{-1}, \dots, x_{n-1}, y_{-1}, \dots, y_{n-1}$. We now will define x_n and y_n .

Assume $(h(n))_2 = 0$. Pick an x such that $x \neq y_i, -1 < i < n - 1$, and,

$$x \in [\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*, q)} \text{ and } \emptyset_{(h(n))_0} \subseteq [\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*, q)}$$

or

$$x \in \emptyset_{(h(n))_0} - ([\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*, q)} \cup \{y_{i-1}\}_{0 < i < n}),$$

and set $y_n = y_{n-1}$ and $x_n = x$.

Otherwise pick a

$$y \in \emptyset_{(h(n))_0} \cap ([\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*, q)} - \text{Int}([\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*, q)}))$$

and

$$x \in [\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*, q)}$$

and set $y_n = y, x_n = x$, where $\text{Int}(X)$ is the interior of the set.

If $(h(n))_2 = 1$ then switch x and y .

This definition is possible because if

$$\emptyset_{(h(n))_0} \not\subseteq [\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*, q)},$$

and

$$\emptyset_{(h(n))_0} - ([\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*, q)} \cup \{y_{i-1}\}_{0 < i < n}) = \emptyset,$$

then $\emptyset_{(h(n))_0} - [\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*, q)}$ is nonempty and finite. Take a $y' \in \emptyset_{(h(n))_0} - [\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*, q)}$. Then $\text{Cl}(y') \cap \emptyset_{(h(n))_0}$ is open, infinite and contains y' . Hence there is a y such that

$$y \in \emptyset_{(h(n))_0} \cap ([\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*, q)} - \text{Int}[\psi_{(h(n))_1}(x)]^{(\mathfrak{A}^*, q)}).$$

Let $\emptyset = \{x_{i-1}\}_{i \in \omega}$ and $q_{\langle a, b \rangle}$ be the topology generated by q, \emptyset , and $\mathbb{N} - \emptyset$. \emptyset and $\mathbb{N} - \emptyset$ are infinite because both of the sets $\{m | (h(n))_2 = 0\}$ and $\{n | (h(n))_2 = 1\}$ are infinite.

Now $a \in \emptyset$ and $b \notin \emptyset$ and each set is infinite so all we need to show is that $(\mathfrak{A}^*, q) <_{\mathcal{Q}} (\mathfrak{A}^*, q_{\langle a, b \rangle})$.

We show this by induction on the number of occurrences of Qx . Since $q \subseteq q_{\langle a, b \rangle}$ we need only to show one direction. So assume that $(\mathfrak{A}^*, q_{\langle a, b \rangle}) \vDash Qx\varphi(x)$ and $(\mathfrak{A}^*, q) \vDash \neg Qx\varphi(x)$ and derive a contradiction. Hence

$$[\varphi(x)]^{(\mathfrak{A}^*, q_{\langle a, b \rangle})} = [\varphi(x)]^{(\mathfrak{A}^*, q)} = (\emptyset_\alpha \cap \emptyset) \cup (\emptyset_\beta \cap \emptyset^c).$$

Either \emptyset_α or \emptyset_β is not a subset of $[\varphi(x)]^{(\mathfrak{A}^*, q)}$ since otherwise $\emptyset_\alpha \cup \emptyset_\beta = [\varphi(x)]^{(\mathfrak{A}^*, q)}$. So assume $\emptyset_\alpha \not\subseteq [\varphi(x)]^{(\mathfrak{A}^*, q)}$ since the other case follows by symmetry.

There are k, l such that $\emptyset_k \subseteq \emptyset_\alpha, \emptyset_k$ basic open, $[\psi_l(x)]^{(\mathfrak{A}^*, q)} = [\varphi(x)]^{(\mathfrak{A}^*, q)}$ and $\emptyset_k \not\subseteq [\psi_l(x)]^{(\mathfrak{A}^*, q)}$. Take $h^{-1}(\langle k, l, o \rangle) = m$ and we have

$$\emptyset_k - ([\psi_l(x)]^{(\mathfrak{A}^*, q)} \cup \{y_{i-1}\}_{0 < i < m}) = \emptyset$$

since otherwise $\emptyset_\alpha \cap \emptyset \not\subseteq [\varphi(x)]^{(\mathfrak{A}^*, q)}$. Thus

$$y_m \in (\emptyset_k \cap [\psi_l(x)]^{(\mathfrak{A}^*, q)} - \text{Int}[\psi_l(x)]^{(\mathfrak{A}^*, q)}).$$

If $\emptyset_\beta = \emptyset$ then we are done since $y_m \notin \emptyset_\alpha \cap \emptyset$.

To finish assume $\emptyset_\beta \neq \emptyset$. Then $y_m \in \emptyset_\beta \cap \emptyset^c$. Hence $y_m \in \emptyset_\beta \cap \emptyset_k \subseteq [\psi_l(x)]^{(\mathfrak{A}^*, q)}$, since if

$$\emptyset_\beta \cap \emptyset_k \cap \{y_{i-1}\}_{0 < i < m} - [\psi_l(x)]^{(\mathfrak{A}^*, q)} \neq \emptyset$$

then $\emptyset_\beta \cap \emptyset_k \cap \emptyset^c - [\psi_l(x)]^{(\mathfrak{A}^*, q)} \neq \emptyset$ which is a contradiction. But y_m is a noninterior point by definition so we have a contradiction.

We have shown the result and we can prove analogously the same result for $\Delta(\mathcal{L}(Q^n)_{n \in \omega})$ via the same method.

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