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AN EXAMPLE CONCERNING INVERSE LIMIT SEQUENCES OF NORMAL SPACES

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ABSTRACT. Using techniques developed by Wage and Przymusiński, we construct an inverse limit sequence (X_n, f_{nm}) with limit space X such that each X_n is Lindelöf with $\dim X_n = 0$, where dim denotes covering dimension, while X is normal with $\dim X > 0$. The space X is a counterexample to several conjectures in Topology.

Let X be the inverse limit of the sequence (X_n, f_{nm}). We recall that if, for some k in N, the set of positive integers, $\dim X_n \leqslant k$ for each n in N, then $\dim X \leqslant k$ if one of the following conditions is satisfied: (a) X_n is perfectly normal for each n in N, (b) X is strongly paracompact, and (c) X is countably paracompact, X_n is normal and f_{nm} is open and surjective for all n, m in N **[1]**, **[4]**.

THE CONSTRUCTION. Let d be the usual metric on the unit interval I. Let ρ be the separable metric on I introduced by Wage **[8]**. The ρ-topology on I is finer than the d-topology, $d \leqslant \rho$ and every ρ-Borel set of I is also d-Borel. Moreover, every nonempty ρ-open set is uncountable and there is an ρ-closed set E such that the boundary of every nonempty ρ-open set disjoint from E has cardinality c, the cardinality of the continuum.

Let $\{A_1, A_2, \cdots\}$ be a partition of I with the property that for every uncountable Borel set of B of I and every n in N, $|B \cap A_n| = c$ **[6**, Theorem 2**]**. We may clearly assume that 0, 1 are in A_1 and that $A_1 \cap E$ is ρ-dense in E. For each x in $I - A_1$ and each n in N, choose x_n^-, x_n^+ in A_2 so that

$$x - \frac{1}{n} < x_n^- < x_{n+1}^- < x < x_{n+1}^+ < x_n^+ < x + \frac{1}{n}.$$

Let $\{(A_\alpha, B_\alpha): \alpha < \omega(c)\}$ be the collection of all pairs of countable subsets of A_1^2 with $|\bar{A}_\alpha^\rho \cap \bar{B}_\alpha^\rho \cap \Delta|$ uncountable, where $\omega(c)$ is the first ordinal of cardinality c and Δ is the diagonal of I^2. For each $\alpha < \omega(c)$ and each n in N, choose x_α in A_1, $(x_{\alpha n}^1, x_{\alpha n}^2)$ in A_α and $(x_{\alpha n}^3, x_{\alpha n}^4)$ in B_α so that $\rho(x_{\alpha n}^i, x_\alpha) \leqslant 1/n$, $x_{\alpha n}^i \lhd x_\alpha$ and $x_\beta \lhd x_\alpha$ for $\beta < \alpha$ and $i = 1, 2, 3, 4$, where $\lhd$ is a well-ordering on I.

For n, m in N and x in I, we define a set $B_m^n(x)$ containing x as follows. For x in $A_2 \cup \cdots \cup A_{n+1} \cup (A_1 - \{x_\alpha: \alpha < \omega(c)\})$, $B_m^n(x) = \{x\}$. For x in $\bigcup_{k=n+2}^\infty A_k$, $B_m^n(x) = [x_m^-, x_m^+]$. On $\{x_\alpha: \alpha < \omega(c)\}$, B_m^n is defined by transfinite induction.

Assuming it has been defined for all $\beta < \alpha$, we set

$$B_m^n(x_\alpha) = \{x_\alpha\} \cup \left(B_{k+2m}^n(x_{\alpha k}^i): k > 2m, i = 1, 2, 3, 4\right).$$

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It is readily seen that $B_m^n(x)$ is a d-closed subset of $[x - 1/m, x + 1/m]$ and $B_m^n(x_\alpha)$ is a countable subset of $\{x\colon x \lhd x_\alpha, \rho(x, x_\alpha) < 1/m\}$. Also $B_{m+1}^n(x) \subset B_m^n(x)$ and if y is in $B_m^n(x)$, then $B_k^n(y) \subset B_m^n(x)$ for some k in N. It follows that $\{B_m^n(x)\colon m \in N\}$ is a local base at x consisting of open-and-closed sets relative to some topology τ_n on I. Next, put $B_m(x) = B_m^n(x)$ if $x = x_\alpha$ for some $\alpha < \omega(c)$, and $B_m(x) = \{x\}$ if not. $B_m(x)$ is a local base of x consisting of open-and-closed sets relative to some topology τ on I. Clearly, τ is finer than the ρ-topology on I, and $\{x_{\alpha n}^i\colon n = 1, 2, \dots\}$ converges to x_α, $i = 1, 2, 3, 4$.

In the sequel, X, X_n denote (I, τ), (I, τ_n), respectively, and $f_{nm}\colon X_m \to X_n$ denotes the identity function. It is readily verified that X is the limit space of the inverse limit sequence (X_n, f_{nm}).

CLAIM 1. *X_n is a T_2 Lindelöf space with* $\dim X_n = 0$.

PROOF. It is obvious that X_n is T_2 and $\operatorname{ind} X_n = 0$. Let $\mathcal{U}$ be an open cover of X_n. Since clearly every point of A_{n+2} has a local base consisting of open intervals, there are d-open sets $G_1, G_2, \dots,$ each contained in some member of $\mathcal{U}$, such that $A_{n+2} \subset G = \bigcup_{i=1}^\infty G_i$. Then, since $(X - G) \cap A_{n+2} = \varnothing$, $X - G$ is countable. It is now clear that $\mathcal{U}$ has a countable open refinement, and hence X_n is Lindelöf. Thus, since also $\operatorname{ind} X_n = 0$, $\dim X_n = 0$.

CLAIM 2. *The family of neighbourhoods of the diagonal of X^2 is a uniformity. Hence X is normal* [2, *Problem* L, *p.* 209].

PROOF. Let G be an open neighbourhood of Δ. It suffices to find a neighbourhood V of Δ with $V \circ V \subset G$.

Let $A = A_1^2 - G$. If $\bar{A}^\rho \cap \Delta$ were uncountable, then, for some $\alpha < \omega(c)$, A would contain $A_\alpha = B_\alpha$ as a countable ρ-dense subset and hence (x_α, x_α) would be a point of A. Thus, $\bar{A}^\rho \cap \Delta$ is countable. Since ρ is separable and X^2 is Tychonoff, there is a cozero cover $\{G_1, G_2, \dots\}$ of the open-and-closed subset A_1 of X such that

$$\Delta \cap A_1^2 \subset \bigcup_{i=1}^\infty G_i^2 \subset G.$$

Let $\{H_1, H_2, \dots\}$ be a cozero star-refinement of $\{G_1, G_2, \dots\}$ and put $V = \bigcup_{i=1}^\infty H_i^2 \cup \Delta$. It is readily verfied that V has the required properties.

CLAIM 3. $\dim X > 0$.

PROOF. Let F be an uncountable ρ-closed set of I with $E \cap F = \varnothing$, and suppose U, V are disjoint open-and-closed sets of A_1 with $E \cap A_1 \subset U$, $F \cap A_1 \subset V$ and $A_1 = U \cup V$. Then V is uncountable and $\bar{U}^\rho \cap \bar{V}^\rho$ is countable, otherwise, for some $\alpha < \omega(c)$, A_α, B_α would be countable ρ-dense subsets of U^2, V^2, respectively, and hence x_α would be in $U \cap V$. Also $\bar{U}^\rho \cup \bar{V}^\rho = \bar{A}_1^\rho = X$ and, since $A_1 \cap E$ is ρ-dense in E, $E \subset \bar{U}^\rho$. Hence $X - \bar{U}^\rho = \bar{V}^\rho - \bar{U}^\rho$ is an uncountable ρ-open set of I contained in $X - E$ with ρ-boundary contained in the countable set $\bar{U}^\rho \cap \bar{V}^\rho$. We conclude that $\dim A_1 > 0$ and hence $\dim X > 0$.

REMARK 1. A space is called N-compact if it is the inverse limit of countable discrete spaces. A zero-dimensional T_2 Lindelöf space is N-compact, and so is the inverse limit of N-compact spaces. Hence X is N-compact although $\dim X > 0$.

Przymusiński's space in [7] has the same property. Spaces exhibiting the same pathology were previously constructed in [3] and [5].

REMARK 2. Kelley [2, p. 209] has conjectured that a topologically complete space with the property that the family of all neighbourhoods of its diagonal is a uniformity is paracompact. X is a counterexample to this conjecture. For if X were paracompact, since it is also locally countable, we would have $\dim X = \operatorname{loc}\dim X = 0$.

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