PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 79, Number 1, May 1980

BOLZANO'S THEOREM IN SEVERAL COMPLEX VARIABLES

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ABSTRACT. Let Ω be a bounded domain in C^n containing the origin. Let $f\colon \bar{\Omega} \to C^n$ be analytic in Ω and continuous in $\bar{\Omega}$, and $\operatorname{Re} \bar{z} \cdot f(z) > 0$ for $z \in \partial\Omega$. It is shown that f has exactly one zero in Ω.

1. The well-known Bolzano theorem [**1**, p. 85] states that if f is a real-valued continuous function on a closed interval $[a, b]$ and $f(a)f(b) < 0$, then f has a zero in (a, b). Without loss of generality we can suppose that $a < 0 < b$ and $f(a) < 0 < f(b)$. Then the condition $f(a)f(b) < 0$ becomes $xf(x) > 0$ for $x \in \partial I$ where ∂I denotes the boundary of the interval $I = (a, b)$. The conclusion is that f has at least one zero in (a, b).

In a recent paper [**5**], the author extends this form of Bolzano's theorem to analytic functions of a complex variable, and indeed, with a stronger conclusion.

THEOREM 1. *Let Ω be a bounded domain of the z plane containing the origin. Let f be analytic in Ω and continuous in $\bar{\Omega}$, and suppose* $\operatorname{Re} \bar{z}f(z) > 0$ *for $z \in \partial\Omega$. Then f has exactly one zero in Ω.*

It is our purpose in the present paper to extend this result to several complex variables. Our approach is an application of degree theory.

2.

THEOREM 2. *Let Ω be a bounded domain in C^n containing the origin. Let $f\colon \bar{\Omega} \to C^n$ be analytic in Ω and continuous in $\bar{\Omega}$, and suppose* $\operatorname{Re} \bar{z} \cdot f(z) > 0$ *for $z \in \partial\Omega$. Then f has exactly one zero in Ω.*

Before we proceed with the proof of Theorem 2, we shall need three lemmas. Lemma 1 may be found in the book of Bochner and Martin [**2**, p. 39].

LEMMA 1. *Let $f = (f_1, \ldots, f_n)\colon D \to C^n$ be analytic in a domain D of C^n. Then the real and complex Jacobians are related by the formula*

$\det(\partial(u_j, v_j)/\partial(x_k, y_k)) = |\det(\partial f_j/\partial z_k)|^2$, *where $z_k = x_k + iy_k$ and $f_j = u_j + iv_j$.*

Observe that $f\colon C^n \to C^n$ can also be considered a map $f\colon R^{2n} \to R^{2n}$. From Lemma 1, we see that the local degree of a complex analytic function at any preimage of a regular value is always $+1$. Thus the number of points in the preimage of a regular value of f is always exactly $\deg f$. In the real case, there can be some cancellation (even for real analytic functions).

Received by the editors May 2, 1979.

AMS (MOS) subject classifications (1970). Primary 32A30.

Key words and phrases. Analytic functions, Bolzano's theorem, homotopy invariance, subvariety.

LEMMA 2. *Let A be an $n \times n$ complex matrix and* $\det A = 0$. *Then for sufficiently small* $\varepsilon > 0$, $\det(A + \varepsilon I) \neq 0$ *where I is the identity matrix.*

PROOF. By Schur's theorem, there is a nonsingular matrix P such that $P^{-1}AP$ is triangular and the leading diagonal elements of $P^{-1}AP$ are eigenvalues of A. Since $\det A = 0$, for small enough $\varepsilon > 0$ we have

$$\det(A + \varepsilon I) = \det\bigl(P^{-1}(A + \varepsilon I)P\bigr) = \det(P^{-1}AP + \varepsilon I) \neq 0.$$

LEMMA 3. *Let U be an open bounded set in C^n and let f, g: $\overline{U} \to C^n$ be two continuous maps. Let $w \in C^n$ and $w \notin f(\partial U) \cup g(\partial U)$. Suppose further that ε satisfies $0 < \varepsilon < \min\{\|f(z) - w\|: z \in \partial U\}$. If $\|f(z) - g(z)\| < \varepsilon$ for all $z \in \partial U$, then* $\deg(w, f, U) = \deg(w, g, U)$.

PROOF. Define the homotopy H: $\overline{U} \times [0, 1] \to C^n$ by

$$H(z, t) := (1 - t)f(z) + tg(z) \quad \text{for } z \in \overline{U} \text{ and } t \in [0, 1].$$

By hypothesis, it is easy to see that $H(z, t) \neq w$ for $z \in \partial U$ and $t \in [0, 1]$. By the homotopy invariance theorem, the result follows.

PROOF OF THEOREM 2. Define the homotopy H: $\overline{\Omega} \times [0, 1] \to C^n$ by

$$H(z, t) := (1 - t)z + tf(z) \quad \text{for } z \in \overline{\Omega} \text{ and } t \in [0, 1].$$

By hypothesis, $H(z, 0) \neq 0$ and $H(z, 1) \neq 0$ for $z \in \partial\Omega$, and $\operatorname{Re} \bar{z} \cdot H(z, t) > \operatorname{Re} \bar{z} \cdot (1 - t)z > 0$ for $z \in \partial\Omega$ and $t \in (0, 1)$. By the homotopy invariance theorem,

$$\deg(0, f, \Omega) = \deg(0, H(z, t), \Omega) = \deg(0, I, \Omega) = 1. \tag{1}$$

Observe that $f^{-1}(0)$ is a compact subvariety of Ω. We shall show that $f^{-1}(0)$ is finite. For this purpose, let M be a component of $f^{-1}(0)$. Then M is a compact subvariety of Ω. Since each coordinate projection $\pi_j(z_1, \dots, z_n) = z_j$ is analytic on M, π_j is constant by applying a result in [3, p. 106]. So M is a singleton; thus $f^{-1}(0)$ is discrete and hence finite. Now, let $\zeta_1, \dots, \zeta_k$ denote the zeros of f. Let Δ_j be a neighborhood of ζ_j such that the closed sets $\overline{\Delta}_j$ are pairwise disjoint and $\overline{\Delta}_j \subset \Omega$. Let $K = \overline{\Omega} - \{\bigcup_1^k \Delta_j\}$, so K is a closed subset of $\overline{\Omega}$ which does not contain a zero of f. By the excision and additivity properties of the degree [4, p. 86], we have

$$\deg(0, f, \Omega) = \deg(0, f, \Omega - K) = \sum_j \deg(0, f, \Delta_j). \tag{2}$$

We claim that $\deg(0, f, \Delta_j) \geqslant 1$ for each j. If $\deg(J_{\zeta_j}(f)) \neq 0$, where $J_{\zeta_j}(f)$ denotes the complex Jacobian matrix of f at ζ_j, then $\deg(0, f, \Delta_j) = 1$ by virtue of Lemma 1. Suppose now that $\det(J_{\zeta_j}(f)) = 0$. Then by Lemma 2, $\det(J_{\zeta_j}(g)) \neq 0$ when $g(z) = \varepsilon(z - \zeta_j) + f(z)$ and $\varepsilon > 0$ is small enough. It therefore follows from Lemmas 1 and 3 that $\deg(0, f, \Delta_j) = \deg(0, g, \Delta_j) \geqslant 1$. By (1) and (2), the theorem is established.

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